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Stability Criteria and the Initial Value Problem for Collisionless, Guiding Center Plasma

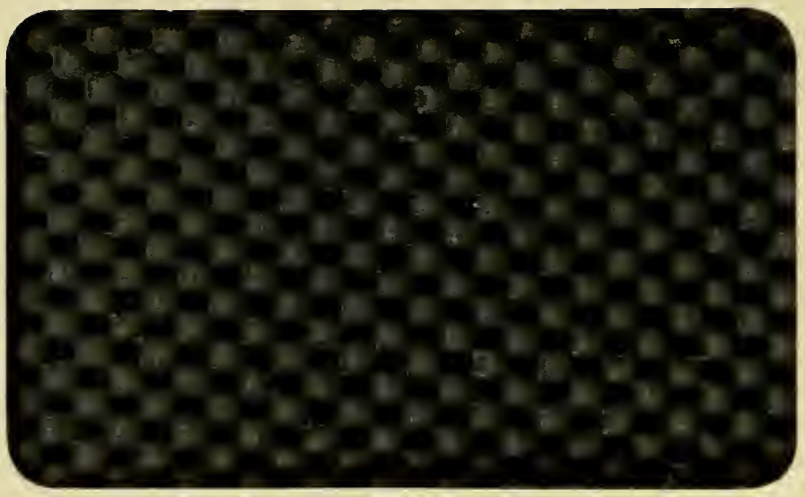
ABRAHAM KADISH

1 April 1965

AEC Research and Development Report

NEW YORK UNIVERSITY

NYC-1480-22 c.2



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Magneto-Fluid Dynamics Division
Courant Institute of Mathematical Sciences
New York University

TID-4500
39th edition

Physics

MF-45

NYO-1480-22

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Contract No. AT(30-1)-1480

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Abstract

Employing the system of integro-differential equations which govern a plasma in the guiding center limit, stability criteria are found for infinite, homogeneous, collisionless, guiding center plasma. A one-dimensional fundamental solution is derived for the linearized system, and a well-posedness theorem is proved for the corresponding initial value problem.

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1. Introduction

The setting for this investigation is a collisionless, charge-neutral plasma, consisting of single species of ions and electrons, permeated by a strong, slowly varying magnetic field. It is assumed that the mass-to-charge ratio, m/e, is small compared to unity for both classes, and that all particle orbits may consequently be described by the lowest order guiding center theory.[1,2,3,4]

Accordingly, the orbits consist of a rapid circular gyration with small Larmor period about a magnetic field line on which is superimposed both a drift velocity which is independent of the particle, and a kinetic velocity parallel to the field. The high frequency gyration induces a local symmetry in the pressure tensor as a result of the circular motion orthogonal to the field lines. As a result, we obtain, in general, an anisotropic stress tensor which depends on two scalar pressures and the orientation of the unit vector, $\vec{\beta}$, in the direction of the magnetic field. Expressed invariantly, the elements of the tensor are given by[5]

$$p_{ij} = p_{\parallel}\beta_i\beta_j + p_{\perp}(\delta_{ij} - \beta_i\beta_j) \tag{1}$$

where β_i are the components of $\vec{\beta}$, and δ_{ij} is the Kronecker delta.

The model we employ is given below, and was derived

by H. Grad.[6,7]

$$(2)\qquad 0 = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\cdot[\vec{u}+v\vec{\beta}]\right)f^{\pm} + \frac{\partial}{\partial v}\left[\left\{\vec{\beta}\cdot\alpha^{\pm}\left(\frac{\operatorname{div}\mathbb{P}^{+}}{\rho^{+}} - \frac{\operatorname{div}\mathbb{P}^{-}}{\rho^{-}}\right)\right.\right.$$

$$\left.\left. + \vec{\beta}\cdot\frac{\partial}{\partial x}\left(\frac{1}{2}u^{2} - \mu B\right) + v(\vec{u}\cdot\vec{\kappa})\right\} f^{\pm}\right] ,$$

$$(3)\qquad 0 = \frac{\partial\vec{B}}{\partial t} + \nabla\times(\vec{B}\times\vec{u}) ,$$

$$(4)\qquad 0 = \rho\vec{B}\times\frac{d\vec{u}}{dt} + \vec{B}\times\operatorname{div}(\mathbb{P}^{+}+\mathbb{P}^{-}) - \frac{1}{\mu_{o}}\vec{B}\times((\vec{\nabla}\times\vec{B})\times\vec{B}).$$

In these equations, all quantities are time-averaged over a Larmor period. The distribution functions, $f^{\pm}$, are taken as functions of the magnetic moment per unit mass, μ, and the kinetic speed, v, parallel to the magnetic field line $\vec{B}$. They are normalized as mass densities in μ-v space in our treatment ($\int f\,d\mu dv = \rho$). The quantities $\mathbb{P}^{\pm}$ and $\rho^{\pm}$ are the pressure tensors and mass densities respectively. (Superscripts $^{+}$ and $^{-}$ will refer to ion and electron quantities.) The macroscopic flow velocity orthogonal to $\vec{B}$ is $\vec{u}$ and $\vec{\kappa}$ represents the curvature of $\vec{B}$. The quantities $\alpha^{\pm}$ are defined by

$$(5)\qquad \alpha^{\pm} = \frac{(m/e)^{\mp}}{(m/e)^{+} + (m/e)^{-}} .$$

Equations (2) are the collisionless Vlasov equations for the

guiding centers of the ions and electrons. Equations (3) are Maxwell's equations for the time-evolution of the magnetic field in which, consistent with the lowest order guiding center theory, the electric field has been replaced by $(\vec{B} \times \vec{u})$. The last equations of the system are the magnetohydrodynamic equations for the components of velocity orthogonal to $\vec{B}$. In these, the current has been replaced by $\frac{1}{\mu_o} (\vec{\nabla} \times \vec{B})$.

In this paper, a space-filling plasma with homogeneous steady state is assumed and our system is linearized about it. The initial value problem for the linearized system is then investigated.

The first two sections exhibit the linearization, and a uniqueness theorem is proven for the purpose of justifying the results of the following section. In that section, which contains the main results of the paper, stability is investigated via Fourier and Laplace transforms. The limitation of the theory to slowly varying fields is manifested in a nondispersive relation which determines the exponentially growing modes of the system, with the consequence that the existence of such modes implies the non-well-posedness[8] of the initial value problem. Following the discussion of stability some results in wave propagation are derived for special limiting values of parameters of the problem. In the last section, a funda-

mental solution for the one-dimensional plasma is derived for arbitrary magnetic field orientation. Some asymptotic properties of this solution are exhibited. In the appendix we treat some of the integrals encountered in the paper.

A word must now be said concerning the stability results which will be derived. In our investigation we employ a method similar to that of O. Penrose,[9] and show that the guiding center theory predicts stability for wide classes of steady-state distributions. In particular, it will be shown that the two necessary conditions for stability which are obtained from the guiding center theory extend those obtained from a purely macroscopic model by R. Lüst,[10] and then are in agreement with those derived in a special case by H. Furth[11] who considered a particular field perturbation and linearized the Vlasov equation about a two-temperature Maxwellian distribution. It is also shown that the aforementioned conditions are not sufficient for stability, but when coupled with a third inequality which is shown to be satisfied by distribution functions that are simply-peaked, the new set is sufficient.

Acknowledgement: The author wishes to take this opportunity to express his indebtedness to Prof. Harold Grad for suggesting this investigation, and to Prof. Harold Weitzner for his many helpful observations and suggestions.

2. Linearization

We now linearize equations (2)-(4) about a homogeneous steady state. Quantities associated with this state will be distinguished by a "o" subscript or superscript. Thus, $\vec{B}_o$ is the magnetic field vector in the steady-state configuration. The absence of such an index on a dependent variable will imply a time-dependent quantity. It is to be assumed that such quantities are sufficiently small (together with their derivatives) to insure that products of them may be neglected.

Our coordinate system is chosen so that the magnetic field vector $\vec{B}_o$ is parallel to the x-axis, and the plasma is at rest (i.e., $\vec{u}_o = (0,0,0)$). Thus,

$$
\begin{aligned}
&\vec{B} = (B_o + B_1,\ B_2,\ B_3) \\
&\vec{u} = (0,\ u_2,\ u_3) \\
&\rho^{\pm} = \rho_o^{\pm} + \rho^{\pm} \\
&f^{\pm} = f_o^{\pm}(\mu,v) + f^{\pm}(\mu,v;\ \vec{x},t) \quad .
\end{aligned} \tag{6}
$$

To first order we have

$$
|B| = \sqrt{(B_o + B_1)^2 + B_2^2 + B_3^2} = B_o + B_1 \quad . \tag{7}
$$

Therefore, the components of $\vec{\beta} = \vec{B}/|B|$ are given to first

order by

$$(8)\qquad \beta_1 = \frac{B_o + B_1}{B_o + B_1} = 1 \ ; \qquad \beta_{2,3} = \frac{B_{2,3}}{B_o + B_1} = \frac{B_{2,3}}{B_o} \ .$$

Thus, using the representation of the pressure tensor as given by (1), we obtain the linearized tensors

$$(9)\qquad P^{\pm} = \begin{pmatrix} p_{\parallel}^{o\pm} + \int v^2 f^{\pm} d\mu dv & -(p_{\perp}^{o\pm} - p_{\parallel}^{o\pm}) \frac{B_2}{B_o} & -(p_{\perp}^{o\pm} - p_{\parallel}^{o\pm}) \frac{B_3}{B_o} \\ -(p_{\perp}^{o\pm} - p_{\parallel}^{o\pm}) \frac{B_2}{B_o} & p_{\perp}^{o\pm}(1 + \frac{B_1}{B_o}) + B_o \int \mu f^{\pm} d\mu dv & 0 \\ -(p_{\perp}^{o\pm} - p_{\parallel}^{o\pm}) \frac{B_3}{B_o} & 0 & p_{\perp}^{o\pm}(1 + \frac{B_1}{B_o}) \\ & & + B_o \int \mu f^{\pm} d\mu dv \end{pmatrix}$$

In deriving (9) we have used the following integral representations of the pressures[12]

$$(10)\qquad \begin{aligned} p_{\parallel}^{\pm} &= \int_0^{\infty} d\mu \int_{-\infty}^{+\infty} v^2 dv \ f^{\pm}(\mu, v) \\ p_{\perp}^{\pm} &= |B| \int_0^{\infty} \mu d\mu \int_{-\infty}^{+\infty} dv \ f^{\pm}(\mu, v) \ . \end{aligned}$$

Substituting these expressions in our system, and keeping only first order terms, we obtain the linear equations

$$(11)\qquad 0 = \left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x}\right) f^{\pm} + \left\{([\eta]\alpha^{\pm} - \mu) f^{\pm}_{o,v} - v\,\frac{f^{\pm}_{o}}{B_o}\right\}\frac{\partial B_1}{\partial x}$$
$$+ \frac{f^{\pm}_{o}}{B_o}\left(\frac{\partial r_2}{\partial x_2} + \frac{\partial r_3}{\partial x_3}\right) + \alpha^{\pm}\, f^{\pm}_{o,v}\,\frac{\partial}{\partial x_1}\int d\mu v^2 dv \left(\frac{f^{+}_{1}}{\rho^{+}_{o}} - \frac{f^{-}}{\rho^{-}_{o}}\right)$$

$$(12)\qquad \begin{cases} 0 = \dfrac{\partial B_1}{\partial t} + \left(\dfrac{\partial r_2}{\partial x_2} + \dfrac{\partial r_3}{\partial x_3}\right) \\ 0 = \dfrac{\partial B_{2,3}}{\partial t} - \dfrac{\partial}{\partial x_1} r_{2,3} \end{cases}$$

$$(13)\qquad \frac{\partial r_{2,3}}{\partial t} - v^2\,\frac{\partial}{\partial x_1} B_{2,3} + c^2\,\frac{\partial B_1}{\partial x_{2,3}} + \frac{B_o^2}{\rho_o}\,\frac{\partial}{\partial x_{2,3}}\int \mu(f^{+} + f^{-})\,d\mu dv$$

in which we have used the abbreviations

$$(14)\qquad \begin{aligned} r_{2,3} &= B_o\, u_{2,3} \\ [\eta] &= \frac{1}{B_o}\left\{\frac{(p_{\perp}^{o+} - p_{\parallel}^{o+})}{\rho_o^{+}} + \frac{(p_{\perp}^{o-} - p_{\parallel}^{o-})}{\rho_o^{-}}\right\} \\ c^2 &= \frac{1}{\rho_o}\left(\frac{B_o^2}{\mu_o} + p_{\perp}^{o}\right) \\ v^2 &= \frac{1}{\rho_o}\left(\frac{B_o^2}{\mu_o} + p_{\perp}^{o} - p_{\parallel}^{o}\right) = c^2 - \frac{p_{\parallel}^{o}}{\rho_o} \end{aligned}$$

In the following the further abbreviations

$$(15)\qquad \begin{aligned} \langle v^2 \rangle &= \iint d\mu dv\; v^2 (f^{+}\rho_o^{+} - f^{-}\rho_o^{-}) \\ \langle \mu \rangle &= \iint \mu d\mu dv (f^{+} + f^{-}) \end{aligned}$$

will be employed.

3. Uniqueness.

The methods of the following section rely very much on a Laplace transform analysis of equations (11)-(13). Such an analysis is limited to solutions which grow no faster than some exponential which is linear in time. However, as G. Backus[8] has noted for some first order differential operators in time, solutions may exist which grow faster than any linear exponential. Thus, a uniqueness theorem is necessary in order to substantiate the results obtained by such a transform analysis for the initial value problem.

It will be assumed that initial data is given which is harmonic in space (i.e., of the form $e^{i\vec{k}\cdot\vec{x}}$) and that the resulting solutions are of the form $\vec{a}(t)e^{ik\cdot x}$. The vector $\vec{a}(t)$ will be shown to be unique. The method of proof will be seen to be applicable to solutions whose Fourier transforms exist for some time $T > 0$. In this case, the time evolution of the Fourier transform would be shown to be unique.

We write

$$\vec{u}(t,\vec{x}) = \vec{u}(t)e^{i\vec{k}\cdot\vec{x}}$$

$$\vec{B}(t,\vec{x}) = \vec{B}(t)e^{i\vec{k}\cdot\vec{x}}$$

$$f^{\pm}(t,\vec{x};\ \mu,v) = f^{\pm}(t;\ \mu,v)e^{i\vec{k}\cdot\vec{x}}$$

and substitute in eq. (11)-(13) to obtain

$$(16)\qquad 0 = \left(\frac{\partial}{\partial t} + ik_1 v\right)f^{\pm} + ik_1 B_1\left\{([\eta]\alpha^{\pm} - \mu)f^{\pm}_{o,v} - v\,\frac{f^{\pm}_o}{B_o}\right\} + i(k_2 r_2 + k_3 r_3)\,\frac{f^{\pm}_o}{B_o} + ik_1\alpha^{\pm}\, f^{\pm}_{o,v}\langle v^2\rangle$$

$$(17)\qquad 0 = \frac{\partial B_1}{\partial t} + i(k_2 r_2 + k_3 r_3)$$

$$0 = \frac{\partial}{\partial t} B_{2,3} - ik_1 r_{2,3}$$

$$(18)\qquad 0 = \frac{\partial}{\partial t} r_{2,3} - ik_1 v^2 B_{2,3} + ik_{2,3}\, c^2 B_1 + \frac{B_o^2}{\rho_o}\, ik_{2,3}\langle\mu\rangle$$

where we have used $\vec{k} = (k_1,k_2,k_3)$, and the quantities $\langle\mu\rangle$ and $\langle v^2\rangle$ appearing here are the same as those of equations (15) modulo the factor $e^{ik\cdot x}$. If we assume two derivatives for the dependent variables, then we may eliminate r_2 and r_3 to obtain, for B_1, the equation

$$(19)\qquad \left(\frac{\partial^2}{\partial t^2} + \left\{k_1^2 v^2 + (k_2^2 + k_3^2)c^2\right\}\right)B_1 = -\,\frac{B_o^2}{\rho_o}\,(k_2^2 + k_3^2)\langle\mu\rangle\ .$$

Equation (19) can also be obtained by integrating equations

(18) with respect to time and using equations (16) and (17) to prove the existence of the second derivative. Solving equation (19) for B_1 in terms of $\langle\mu\rangle$. we have

$$(20) \qquad B_1 = b_1 e^{i(\Omega_k t)} + b_2 e^{-i(\Omega_k t)} - \frac{B_o^2}{\rho_o} \frac{(k_2^2 + k_3^2)}{\Omega_k} \int_o^t d\tau \sin[\Omega_k(t-\tau)]\langle\mu(\tau)\rangle$$

where we have written

$$(21) \qquad \Omega_k = \left\{ \nu^2 k_1^2 + (k_2^2 + k_3^2)c^2 \right\}^{1/2}$$

and b_1 and b_2 are homogeneous of degree one in $B_1(0)$ and $\partial B_1/\partial t\Big|_{t=0}$, the latter being given by the initial data of $r_2(0)$ and $r_3(0)$. In a later section, it will be shown that if $\nu^2 < 0$, the initial value problem is not well set for the linear system. Hence, we consider only the case $\nu^2 > 0$, which guarantees that Ω_k is real for real $\vec{k}$. It should be noted that if Ω_k is not real, the proof of uniqueness is still valid with minor modifications in the estimates which will be made. However, the estimates of growth in time are very different. The reality of Ω_k will be seen to yield linear exponential bounds on the functions, whereas the well-posedness theorem for $\nu^2 < 0$ will be shown to imply otherwise. That is, in this case, the problem cannot be solved by Laplace transforms in the space of solutions having Fourier

transforms in space.

Returning to the proof, we replace the expression $i(k_2r_2 + k_3r_3)$ in eq. (16) by $-\partial B_1/\partial t$ from eq. (17), thereby obtaining equations for $f^{\pm}$ in terms of B_1. But B_1 is known as a functional of $f^{\pm}$, and, therefore, we obtain from eq. (16) expressions of the form

$$(21) \qquad \left(\frac{\partial}{\partial t} + ik_1v\right)f^{\pm} = G^{\pm}(f^{+}(\tau),f^{-}(\tau))$$

where $G(f^{+},f^{-})$ is a linear functional of the vector (f^{+},f^{-}) and is composed of integrals thereof. We write eq. (21) in the equivalent form

$$(22) \qquad f^{\pm}(t;\mu,v) = f^{\pm}(0;\mu,v)e^{-ik_1vt} + \int_0^t e^{-ik_1v(t-\tau)}G^{\pm}(f^{+}(\tau),f^{-}(\tau))d\tau$$

and have, upon expanding G under the integral sign, the lengthy expression

$$(23) \qquad f^{\pm}(t) = f^{\pm}(0)e^{-ik_1vt} + \int_0^t ik_1\alpha^{\pm}f^{\pm}_{o,v}\langle v^2(\tau)\rangle e^{-ik_1v(t-\tau)}dt$$
$$+ \int_0^t d\tau\, e^{-ik_1v(t-\tau)}\left[i\Omega_k \frac{f^{\pm}_o}{B_o}(b_1e^{i\Omega_k\tau} - b_2e^{-i\Omega_k\tau})\right]$$
$$+ \int_0^t d\tau\, e^{-ik_1v(t-\tau)}\left[-ik_1\left\{([\eta]\alpha^{\pm} - \mu)f^{\pm}_{o,v} - v\frac{f^{\pm}_o}{B_o}\right\}(b_1e^{i\Omega_k\tau} + b_2e^{-i\Omega_k\tau})\right]$$

(23)

$$+\int_0^t d\tau\, e^{-ik_1 v(t-\tau)}\Big[ik_1\Big\{([\eta]\alpha^{\pm}-\mu)f^{\pm}_{o,v}-v\,\frac{f^{\pm}_o}{B_o}\Big\}\frac{B_o^2}{\rho_o}\,\frac{(k_2^2+k_3^2)}{\Omega_k}$$

$$\times\int_0^\tau d\xi\,\sin[\Omega_k(\tau-\xi)]\langle\mu(\xi)\rangle\Big]$$

$$+\int_0^t d\tau\, e^{-ik_1 v(t-\tau)}\Big[-\frac{f^{\pm}_o}{\rho_o}B_o(k_2^2+k_3^2)\int_0^\tau d\xi\,\cos[\Omega_k(\tau-\xi)]\langle\mu(\xi)\rangle\Big].$$

We now reproduce the quantities $\langle\mu(t)\rangle$ and $\langle v^2(t)\rangle$ from eq. (23) by first multiplying by μ the equations for f^+ and f^- and adding, and then multiplying by $v^2(1/\rho_o^+)$ and $v^2(1/\rho_o^-)$ respectively and subtracting. Integrations with respect to μ and v then yield expressions of the type

(24)
$$|\langle\mu(t)\rangle| \le P_2^{(\mu)}(t) + a_\mu\int_0^t d\zeta\int_0^\zeta |\langle v^2(\tau)\rangle|d\tau$$

$$+ b_\mu\int_0^t d\zeta\int_0^\zeta |\langle\mu(\tau)\rangle|d\tau$$

$$+ c_\mu\int_0^t d\zeta\int_0^\zeta d\tau\int_0^\tau |\langle\mu(\xi)\rangle|d\xi$$

(25)
$$|\langle v^2(t)\rangle| \le P_2^{(v^2)}(t) + a_{v^2}\int_0^t d\zeta\int_0^\zeta |\langle v^2(\tau)\rangle|d\tau$$

$$+ b_{v^2}\int_0^t d\tau\int_0^\tau |\langle\mu(\xi)\rangle|d\xi$$

$$+ c_{v^2}\int_0^t d\zeta\int_0^\zeta d\tau\int_0^\tau |\langle\mu(\xi)\rangle|d\xi \quad .$$

In obtaining (24) and (25) the exponentials in (23) were majorized in absolute value by unity. The quantities a_μ, b_μ, c_μ, a_{v^2}, b_{v^2}, and c_{v^2} are maximal estimates for the integrals of $f_o^{\pm}$ and $f_{o,v}^{\pm}$ with respect to the moments which occur together with the coefficients of equation (23) which multiply them. The expressions $P_2^{(\mu)}(t)$ and $P_2^{(v^2)}(t)$ are second-degree polynomials which are positive and are homogeneous of degree one in $f^{\pm}(0;\mu,v)$, b_1 and b_2.

We define the quantity $m = \max\left\{a_\mu, b_\mu, c_\mu, a_{v^2}, b_{v^2}, c_{v^2}\right\}$ and the functions $X(t)$ and $Y(t)$ as

$$
\begin{aligned}
X(t) &= \int_o^t d\zeta \int_o^\zeta d\tau \int_o^\tau |\langle \mu(\xi) \rangle| d\xi \\
Y(t) &= \int_o^t d\zeta \int_o^\zeta d\tau \; |\langle v^2(\tau) \rangle|
\end{aligned}
\tag{26}
$$

together with the functions $R(t)$ and $S(t)$ as

$$
\begin{aligned}
R(t) &= \frac{\partial^3 X}{\partial t^3} - m\left\{Y(t) + \left(\frac{\partial}{\partial t} + 1\right)X(t)\right\} \\
S(t) &= \frac{\partial^2 Y}{\partial t^2} - m\left\{Y(t) + \left(\frac{\partial}{\partial t} + 1\right)X(t)\right\} .
\end{aligned}
\tag{27}
$$

Then, in view of the definitions of $X(t)$, $Y(t)$, and m, reference to equations (24) and (25) yields that

$$
\begin{aligned}
|R(t)| &\leq P_2^{(\mu)}(t) \\
|S(t)| &\leq P_2^{(v^2)}(t) .
\end{aligned}
\tag{28}
$$

We now solve equations (27) and use (28) to obtain estimates for the functions $X(t)$, $Y(t)$ and their derivatives. These, in turn, will yield bounds on $|\langle \mu(t) \rangle|$ and $|\langle v^2(t) \rangle|$ via inequalities (24) and (25). We reduce (27) to a first-order system by defining the vector

$$(29) \qquad F(t) = \left(\frac{\partial^2 X}{\partial t^2}, \frac{\partial X}{\partial t}, X, \frac{\partial Y}{\partial t}, Y \right)$$

and obtain

$$(30) \qquad \frac{\partial}{\partial t} F(t) - m \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ \frac{1}{m} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{m} & 0 \end{pmatrix} F(t) = \begin{pmatrix} R(t) \\ 0 \\ 0 \\ S(t) \\ 0 \end{pmatrix}$$

together with the initial data

$$(31) \qquad \vec{F}(0) = 0 \ .$$

From eq. (30) and (31) we easily obtain that

$$(32) \qquad F_i(t) < (3m+1) \int_0^t \sum_{i=1}^{5} F_i(\tau) d\tau + \int_0^t \left\{ R(\tau) + S(\tau) \right\} d\tau$$

$$i = (1,2,3,4,5)$$

or

$$F_i(t) < \sum F_i(t) < 5(3m+1) \int_0^t \sum_{i=1}^{5} F_i(\tau)d\tau + 5 \int_0^t (R(\tau) + S(\tau))d\tau. \tag{33}$$

Defining

$$H(t) = \int_0^t \sum_{i=1}^{5} F_i(\tau)d\tau \tag{34}$$

we have

$$\begin{aligned} &\frac{dH}{dt} - 5(3m+1)H(t) < 5 \int_0^t (T(\tau) + S(\tau))d\tau \\ &H(0) = 0 \end{aligned} \tag{35}$$

which implies

$$H(t) < 5 \int_0^t e^{5(3m+1)(t-\tau)} \int_0^\tau \left\{ R(\xi) + S(\xi) \right\} d\xi \ . \tag{36}$$

Substituting in equation (32) and using the bounds from (28) we finally obtain

$$F_i(t) < 5(3m+1) \int_0^t e^{5(3m+1)(t-\tau)} \int_0^\tau \left\{ P_2^{(\mu)}(\xi) + P_2^{(v^2)}(\xi) \right\} d\xi. \tag{37}$$

Since $P_2^{(\mu)}(\xi)$ and $P_2^{(v^2)}(\xi)$ depend linearly and homogeneously

on the initial data, this yields the uniqueness of the $F_i(t)$, and, consequently, of $|\langle\mu(t)\rangle|$ and $|\langle v^2(t)\rangle|$ via eq. (24) and (25). The polynomial bound on $P_2^{(\mu)}$ and $P_2^{(v^2)}$ in turn implies a linear exponential bound on the F_i and, as a result, on $|\langle\mu(t)\rangle|$ and $|\langle v^2(t)\rangle|$. Using this bound on $|\langle\mu(t)\rangle|$. equation (20) yields a similar bound on $B_1(t)$ together with uniqueness. If we integrate equations (17) and (18) and eliminate $r_{2,3}$, we obtain the necessary estimates for $B_{2,3}$. Eliminating $B_{2,3}$, we obtain the estimates for $r_{2,3}$.

To recapitulate, we have shown that if $f_o^{\pm}$ and $f_{o,v}^{\pm}$ decay sufficiently fast so that the moments which appear are absolutely integrable. then solutions which are harmonic in space, or which have Fourier transforms for some time $T > 0$, are unique within the class of such functions. Moreover, if $\frac{1}{\rho_o}(B_o^2/\mu_o + P_\perp^o - P_{\shortparallel}^o) > 0$, then the time rate of growth of these solutions, on the transforms, is bounded by a linear exponential function of time which is a function of the wave vector or of the transform vector.

IV. Stability

A. The Transformed Equations and a Well-posedness Theorem.

Before proceeding to an analysis of wave propagation and stability, we will prove a well-posedness theorem for the initial value problem associated with the system of equations (11)-(13).

We say that the problem is well posed if and only if a unique solution exists for the initial data and, moreover, the solution depends continuously on the data. That is, if $s(t;\vec{I}(\vec{x}))$ is the solution corresponding to the initial data $\vec{I}(\vec{x})$, then we require that for every $\varepsilon > 0$ there should exist a $t_o > 0$ and a $\delta > 0$, such that if $|\vec{I}(\vec{x})| < \delta$ and $t < t_o$, then $|\vec{s}(t;I(\vec{x}))| < \varepsilon$.

The theorem we prove is the following: if initial data is given which is harmonic in space with wave vector $\vec{k}$, and if, corresponding to $\vec{k}$, there are solutions of the form $e^{-i(\omega(\vec{k})t - i\vec{k}\cdot\vec{x})}$ with $\mathrm{Im}(\omega(\vec{k})) > 0$, then the initial value problem for our system is not well posed.

Suppose that such $\omega(\vec{k})$ existed, then substitution of the solution into the system yields an algebraic system which is homogeneous of degree one in the vector $\tilde{\vec{k}} = (\omega(\vec{k}),\vec{k})$. Thus, if we replace $\vec{k}$ by $\lambda^2\vec{k}$ and $\omega(\vec{k})$ by $\lambda^2\omega(\vec{k})$, then the equations are still satisfied. Now suppose that an eigenvector of the system associated with $\omega(\vec{k})$ is the initial data amplitude vector $\vec{v}_o$, some of whose components of course

contain the parameters μ and v. Then if we associate with $\lambda^2\vec{k}$ the initial data vector $\vec{v}_o/\lambda^2$ and the frequency $\lambda^2\omega(\vec{k})$ and consider the limit as $\lambda^2 \to +\infty$, we obtain initial data which is arbitrarily small, but solutions which are arbitrarily large for any finite time. This completes the proof.

In particular, if we take the plane of the vector $\vec{k}$ to be the x-y plane, and the initial data vector such that all dependent variables are zero except r_3 and B_3, then the system becomes

$$
\begin{aligned}
0 &= \frac{\partial B_3}{\partial t} - ik_1 r_3 \Longrightarrow 0 = \omega B_3 - k_1 r_1 \\
0 &= \frac{\partial r_3}{\partial t} - ik_1 v^2 B_3 \Longrightarrow 0 = \omega r_3 - k_1 v^2 B_3 \, .
\end{aligned}
\tag{38}
$$

so that if $k_1 \neq 0$, $\omega(\vec{k}) = \pm k_1 v$ sufficies for the existence of non-trivial solutions. Therefore, the positivity of v^2 is necessary for well posedness and exponential stability. In passing, we note that this inequality ($v^2 > 0$) was obtained by Lüst[10] as a condition for stability from a macroscopic plasma model.

As a result of the above conclusions, we will assume in the following that $v^2 > 0$, and use the results of the uniqueness theorem to solve the system of equations (16)-(18) by Laplace transform analysis. If p is the transform variable, we obtain

$$(38)\qquad \left(\frac{p}{ik_1}+v\right)\hat{f}^{\pm}+\hat{B}_1\left\{([\eta]\alpha^{\pm}-\mu)f^{\pm}_{o,v}-v\,\frac{f^{\pm}_o}{B_o}\right\}$$

$$+\left(\frac{k_2}{k_1}\hat{r}_2+\frac{k_3}{k_1}\hat{r}_3\right)\frac{f^{\pm}_o}{B_o}+\alpha^{\pm}f^{\pm}_{o,v}\langle\hat{v}^2\rangle=\frac{f^{\pm}(0)}{ik_1}$$

$$(39)\qquad \frac{p}{ik_1}\hat{B}_1+\left(\frac{k_2}{k_1}\hat{r}_2+\frac{k_3}{k_1}\hat{r}_3\right)=\frac{B_1(0)}{ik_1}$$

$$\frac{p}{ik_1}\hat{B}_{2,3}-\hat{r}_{2,3}=\frac{B_{2,3}(0)}{ik_1}$$

$$(40)\qquad \frac{p}{ik_1}\hat{r}_{2,3}-\nu^2\hat{B}_{2,3}+\frac{k_{2,3}}{k_1}c^2\hat{B}_1+\frac{B_o^2}{\rho_o}\frac{k_{2,3}}{k_1}\langle\hat{\mu}\rangle=\frac{r_{2,3}(0)}{ik_1}$$

where we denote

$$(41)\qquad \hat{Q}(p)=\int_o^{\infty}e^{-pt}\,Q(t)dt$$

if we are referring to solutions of the form $Q(t)e^{i\vec{k}\cdot\vec{x}}$, or

$$(42)\qquad \hat{Q}(p,\vec{k})=\int_o^{\infty}e^{-pt}\,\tilde{Q}(t.\vec{k})dt$$

if $\tilde{Q}(t,\vec{k})$ is the Fourier transform of $Q(t,x)$.

In our discussion of stability, it will suffice to keep the vector $\vec{k}$ fixed. If it is a Fourier transform variable, then all remarks pertaining to the "solution" of our discussion will be applicable to the "Fourier transforms".

Thus, we now take the plane of $\vec{k}$ to be the x-y plane.

The wave vector $\vec{k}$ is assumed to make an angle θ, not equal to $\pm \pi/2$ with the vector $\vec{B}_o$. (The special case $\theta = \pi/2$ will be considered separately and will be solved exactly.) After defining the variables $z = -p/ik_1$ and $\lambda = \tan \theta$, we have

$$(43) \qquad (v-z)\hat{f}^{\pm} + \hat{B}_1\left\{([\eta]\alpha^{\pm} - \mu)f^{\pm}_{o,v}\right\}(v-z)\frac{\hat{B}_1}{B_o}$$

$$+ \alpha^{\pm} f^{\pm}_{o,v}\langle\hat{v}^2\rangle = \frac{1}{ik_1}\left(f^{\pm}(0;\mu,v) - \frac{B_1(0)}{B_o} f^{\pm}_o\right)$$

$$(44) \qquad -z\hat{B}_1 + \lambda\hat{r}_2 = \frac{B_1(0)}{ik_1}$$

$$-z\hat{B}_{2,3} - r_{2,3} = \frac{B_{2,3}(0)}{ik_1}$$

$$(45) \qquad -z\hat{r}_2 - v^2\hat{B}_2 + \lambda c^2\hat{B}_1 + \frac{B_o^2}{\rho_o}\lambda\langle\hat{\mu}\rangle = \frac{r_2(0)}{ik_1}$$

$$-z\hat{r}_3 - v^2\hat{B}_3 = \frac{r_3(0)}{ik_1} \quad .$$

Dividing eq. (43) by (v-z) and multiplying by the appropriate kernels (i.e., $v^2/\rho_o^{\pm}$ or μ) and combining to reproduce the quantities $\langle\hat{\mu}\rangle$ and $\langle\hat{v}^2\rangle$, we obtain the following system for the vector $\hat{s}(z) = (\langle\hat{\mu}\rangle, \langle\hat{v}^2\rangle, \hat{B}_1, \hat{B}_2, \hat{r}_2, \hat{B}_3, \hat{r}_3)$:

$$(46) \qquad A(z;\lambda)\hat{s}(z) = \frac{1}{ik_1}\vec{j}(0) \ ,$$

where $\overline{j}(0)$ is a vector given by

$$(47)\qquad \vec{j}(0) = \begin{pmatrix} \int \mu \dfrac{g^+(0)+g^-(0)}{v-z}\, d\mu dv \\ \int v^2 \dfrac{g^+(0)/\rho_o^+ - g^-(0)/\rho_o^-}{v-z} d\mu dv \\ B_1(0) \\ B_2(0) \\ r_2(0) \\ B_3(0) \\ r_3(0) \end{pmatrix}; \quad g^{\pm}(0) = f^{\pm}(0) - \frac{B_1(0)}{B_o} f_o^{\pm}$$

and $A(z;\lambda)$ is a 7 x 7 matrix:

$$(48)\qquad A = \begin{pmatrix} 1 & -I_\mu(|\alpha| f_{o,v})_\Delta & -\left\{\left(\dfrac{p_\perp^o}{B_o^2} + I_\mu 2 (f_{o,v})_\Sigma + [\eta] I_\mu(|\alpha| f_{o,v})_\Delta\right\}\right. & 0 & 0 & 0 & 0 \\ 0 & T & \left\{[\eta]T - z^2 I_\mu\left(\dfrac{f_{o,v}}{\rho_o}\right)\right\} & 0 & 0 & 0 & 0 \\ 0 & 0 & -z & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & -z & -1 & 0 & 0 \\ \dfrac{\lambda B_o^2}{\rho_o} & 0 & \lambda c^2 & -\nu^2 & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & -1 \\ 0 & 0 & 0 & 0 & 0 & -\nu^2 & -z \end{pmatrix}$$

in which the following notation has been employed:

$$I_{(\mu^k)}(h) = \int_0^{\infty} d\mu \int_{-\infty}^{+\infty} \frac{\mu^k h(\mu,\nu)}{\nu - z}\, d\nu$$

(+5') $\qquad q_\Delta = q^+ - q^- : \qquad q_\Sigma = q^+ + q^-$

$$T = (1 + |\alpha|_\Sigma) - z^2 I_{(\mu^2)}(|\alpha| f_{0,\nu}/\rho_0)_\Sigma \quad .$$

A calculation yields that

(+9) $\qquad \det A(z;\lambda) = - z(z^2 - \nu^2) D(z;\lambda^2)$

with $D(z;\lambda^2)$ given by

(50) $\qquad D(z;\lambda^2) = T(z^2 - [\nu^2 + \lambda^2(c^- + \frac{p_\perp^2}{\rho_0})] - \lambda^2 \frac{B_0^2}{\rho_0} I_{\mu^2}(f_{0,\nu})_\Sigma)$

$$- z^2\lambda^2 \frac{B_0^2}{\rho_0} (\alpha|\rho)^{\pm} \left(I_\mu(f_{0,\nu}/\rho_0)_\Delta \right)^2 \quad .$$

We note that A may be written as

(+1') $\qquad A = \begin{pmatrix} M & & \\ & -z & -1 \\ & -\nu^2 & -z \end{pmatrix}$

with M a 5 x 5 matrix with determinant

(51) $$\det M = - zD(z;\lambda^2) .$$

If we assume the Hölder continuity and absolute integrability of $\int_o^\infty \mu^k f^{\pm}_{o,v}\, d\mu$ $(k = 0,1,2)$, then all the elements of $A(z;\lambda)$ are analytic in z in a half plane and continuous up to the real line.[13,14] We note that, in general, the elements cannot be continued analytically across this line, and, therefore, if $\vec{k}$ is a transform variable it must be restricted to a half space, $k_1 < 0$ or $k_1 > 0$. The qualitative results for stability are the same in both cases, since if p yields an instability for $k_1 > 0$, -p yields the same result for $-k_1$, as an inspection of $D(z;\lambda^2)$ readily reveals. Thus, we take, without loss of generality, $k_1 > 0$ and our criterion for stability becomes, assuming $\nu^2 > 0$, that $D(z;\lambda^2)$ shall have no zeros in the upper half of the z-plane, which is its domain of analyticity (assuming the absolute integrability of the quantities mentioned above). It actually suffices that they belong to L_p, $p \geq 1$.

Before proceeding to the investigation of stability we note that had we sought plane wave solutions for a fixed wave vector. the complex frequencies, ω, of propagation would have appeared as roots of the determinant of the matrix A, with $z = \pm\, \omega/k_1$. However. it would have been impossible to simply discover real roots of $D(z;\lambda^2)$ since these would give rise to improper integrals (i.e., v-z would vanish for some v in the range of integration) and a

special interpretation of such integrals would have been necessary.[15] An evaluation of the inverse Laplace transform by integration along a path in the upper half plane avoids this problem, and in yielding the complete solution to the problem uncovers all such modes of propagation.

B. Stability Criteria

In the following derivation of stability criteria for the steady state configuration, we follow O. Penrose[9] and exploit the principle of the argument for analytic functions.[16] The statement of the theorem we employ is the following: If $f(z)$ is analytic in a simply connected domain $\mathcal{D}$ and continuous up to its boundary $\partial\mathcal{D}$, then the number of zeros of $f(z)$ in $\mathcal{D}$ is equal to the index of the image of $\partial\mathcal{D}$ with respect to zero when $\partial\mathcal{D}$ is taken with $\mathcal{D}$ to its left in the z-plane.

With a view toward the application of the theorem, we consider functions of the form

$$f(z) = z^2 a(z) + b(z) \tag{52}$$

with $a(z)$ and $b(z)$ analytic and bounded in the upper half plane and continuous up to the real like. We further require

that for z belonging to this half plane

$$\lim_{|z| \to \infty} a(z) = a = \text{real positive number}$$

$$\lim_{|z| \to \infty} b(z) = b = \text{real number.} \tag{53}$$

Also, for z real

$$a(z) = \overline{a(-\bar{z})}$$

$$b(z) = \overline{b(-\bar{z})} \tag{54}$$

where the $\bar{\xi}$ denotes the ξ-conjugate.

We now consider the mapping of the domain bounded by the curves $C_1: z = Re^{i\phi}$; $0 \leq \phi \leq \pi$ and $C_2: z = x + i0$; $-R \leq x \leq +R$ with $R >> 1$ (see Fig. 1). As a result of eq. (53), C_1 is mapped into a large arc, C_1', which approximates a circle crossing the real axis in the image plane near $-\infty$. As ϕ moves through increasing values, the direction taken by C_1' is counter-clockwise (see Figs. 2, 3, 4). As a result of the conditions imposed on a(z) and b(z) in eq. (54) we are assured that the image, C_2', of C_2 crosses the real axis when z = 0, and the image for z negative is the conjugate of that for z positive (i.e., they are reflections of each other in the real axis of the image plane. However, the direction vectors of the tangents are images of each other in the imaginary axis of the plane.)

Figs. 1-4 are chosen to illustrate the nature of the possibilities that may arise and to motivate the derivation of the stability criteria.

Fig. 1 simply illustrates the domain $\mathcal{D}$ (striped area) and locates some points which are symmetrically placed on the real axis, so that the direction taken by the mapping may be followed easily.

In Figs. 2-4, the vertical axes are not drawn so that the nature of the mappings may be distinguished when $f(0) > 0$ and $f(0) < 0$.

In Fig. 2 we have the simplest mapping. It is one-to-one with the image of the real axis crossing the real axis in the image domain only at $z = 0$. We note that if $f(0) < 0$ the index of the image of ∂D with respect to zero is zero, and if $f(0) > 0$ the index is one. Hence, in this case, if $f(0) < 0$ then $f(z)$ does not take on the value zero in the upper half plane, but if $f(0) > 0$ it assumes this value once.

Fig. 3 illustrates a slightly more complicated mapping. Here the image of C_2 crosses the real axis at $z = \pm x_1$ in addition to $z = 0$ to yield a mapping which is not one-to-one. We note that if $f(0) > 0$ then the index of the boundary image with respect to the origin is one, and $f(z)$ assumes the value zero just once in the half-plane Im $x > 0$. If $f(0) < 0$ and $f(\pm x_1) > 0$ then the index with respect to the origin is two and $f(z)$ assumes the value zero twice in this half plane. When $f(0) < 0$ and $f(\pm x_1) < 0$ we have an index of zero with

respect to the origin, so that $f(z)$ is never equal to zero in the upper half plane.

The inference we are led to by the above considerations is that when $f(0) < 0$, $f(z)$ will take the value zero $2n$ times ($n = 0,1,2,\ldots$), but when $f(0) > 0$ it will assume this value $2n+1$ times (for the same range of n). We now prove this theorem. Since $f(\pm \infty)$ are real and positive and $f(0)$ is real, the images of the half lines $x + i0 > 0$ and $x + i0 < 0$ induce an index with respect to the origin of $\frac{1}{2}n$ ($n=-1,0,1,2,\ldots$). If $f(0) < 0$ then n is odd. If $f(0) > 0$ then n is even. As a result of the symmetry of the images of these two half lines, these indices are the same and supplementary. That is, if $f(0) < 0$ then the index due to the real line is $2 \times \frac{1}{2}n$ with n odd, while if $f(0) > 0$ then the index is $2 \times \frac{1}{2}n$ with n even. Thus, adding the index one due to the image of C_1, we have proven the theorem.

From the arguments preceding the theorem, one might be tempted to conclude that any additional crossings of the real axis in the image plane must also occur at negative values. However, as is shown in Fig. 4, this is not the case. Here we see that if $f(0) < 0$ together with both $f(\pm x_1) > 0$ and $f(\pm x_2) > 0$, then the index of the image of ∂D with respect to the origin is still zero, so that $f(z)$ has no zeros in D. It is obvious that such a condition together with $f(0) < 0$ would, however, be sufficient for $f(z)$ not to have zeros in the half plane.

We now return to the function $D(z;\lambda^2)$ and impose the conditions that $f_o^{\pm}(\mu,v)$ should be even in v, and, in particular, that they should be functions of v^2. Then we have, together with the conditions imposed on a(z) and b(z) above, that the remarks made pertaining to zeros of f(z) above are applicable to $D(z;\lambda^2)$ (see Appendix). However, exponential instability is equivalent to the existence of zeros of $D(z;\lambda^2)$ in the upper half plane of z, assuming $\nu^2 > 0$, and, hence, for exponential stability of the plasma, we must have

$$(55)\quad 0 > D(0;\lambda^2) = -(1+|\alpha|_\Sigma)\left\{\nu^2+\lambda^2\left(c^2+\frac{p_\perp^o}{\rho_o}+\frac{B_o^2}{\rho_o}\,\text{P.V.}\right.\right.$$
$$\left.\left.\times\int \mu^2\,\frac{(f_{o,v})_\Sigma}{v}\,d\mu dv\right)\right\}$$

where P.V. denotes principle value. With the hypothesis $f_o^{\pm}$ are functions of v^2 rather than v; the principle value is not necessary since

$$(56)\quad \text{P.V.}\int \frac{\mu^2}{v}\,(f_{o,v})_\Sigma\,d\mu dv = 2\int \mu^2(f_{o,v^2})d\mu dv\ .$$

Since $\nu^2 > 0$ and we want $D(0;\lambda^2) < 0$ for all finite λ^2, eq. (55) becomes

$$(57)\quad c^2+\frac{p_\perp^o}{\rho_o}+\frac{B_o^2}{\rho_o}\,2\int_o^\infty d\mu\int_{-\infty}^{+\infty} dv\;\mu^2(f_{o,v^2})_\Sigma > 0$$

$$c^2 = \frac{1}{\rho_o}\left(\frac{B_o^2}{\mu_o}+p_\perp^o\right)$$

Eq. (57) presents us with a relatively simple first test for the stability of the plasma. In particular, it is simple to show that all spherically symmetric functions (i.e., functions of the form $f_o^{\pm} = f_o^{\pm}(\mu B_o + \frac{1}{2}v^2)$) satisfy the inequality. For such functions we have

$$f^{\pm}_{o,v^2} = \frac{1}{2B_o} f^{\pm}_{o,\mu} \quad . \tag{58}$$

Substituting eq.(58) into (57) and integrating by parts with respect to μ yields

$$c^2 + \frac{p_\perp^o}{\rho_o} - \frac{2}{\rho_o} B_o \int_o^\infty \mu d\mu \int_{-\infty}^{+\infty} (f_o)_\Sigma dv = c^2 - \frac{p_\perp^o}{\rho_o} = \frac{1}{\rho_o} \frac{B_o^2}{\mu_o} > 0. \tag{59}$$

We note that for such functions $p_\perp^{o\pm} = p_{\shortparallel}^{o\pm}$,† so that $v^2 = \frac{1}{\rho_o} B_o^2/\mu_o$ and the two necessary conditions for stability degenerate to a single one.

While eq. (57) presents us with a simply verifiable necessary condition for stability, something more than this may be obtained from it by introducing dimensionless variables. Thus, we write

$$f_o^{\pm}(\mu, v^2) = a_{\pm}\, g_o^{\pm}(b_{\pm}\mu,\ c_{\pm}^2 v^2) \tag{60}$$

and evaluate $a_{\pm}$, $b_{\pm}$, and $c_{\pm}^2$ in terms of $\rho_o^{\pm}$, $p_\perp^{o\pm}$, and $p_{\shortparallel}^{o\pm}$. Doing so we obtain

† See appendix.

$$c_{\pm}^2 = \frac{\rho_o^{\pm}}{p_{\shortparallel}^{o\pm}} \frac{\iint x^2 g_o^{\pm}(y,x^2)\,dydx}{\iint g_o^{\pm}(y,x^2)\,dydx}$$

(61)
$$b_{\pm} = B_o \frac{\rho_o^{\pm}}{p_{\perp}^{o\pm}} \frac{\iint y\, g_o^{\pm}(y,x^2)\,dydx}{\iint g_o^{\pm}(y,x^2)\,dydx}$$

$$a_{\pm} = \frac{B_o(\rho_o^{\pm})^{5/2}}{p_{\perp}^{o\pm}(p_{\shortparallel}^{o\pm})^{\frac{1}{2}}} \frac{(\iint x^2 g_o^{\pm}(y,x^2)\,dydx)^{\frac{1}{2}} \iint y\, g_o^{\pm}(y,x^2)\,dydx}{(\iint g_o^{\pm}(y,x^2)\,dydx)^{5/2}}$$

Substituting $y = b_{\pm}\mu$: $c_{\pm}^2 v^2 = x^2$ in eq. (57) we obtain

(62)
$$0 < \frac{1}{\rho_o}\left\{\frac{B_o^2}{\mu_o} + 2p_{\perp}^o + 2\left(\frac{(p_{\perp}^o)^2}{p_{\shortparallel}^o} \frac{\int x^2 g\,dydx \int y^2 \frac{\partial g}{\partial x^2}\,dydx}{(\int y\, g\,dydx)^2}\right)_{\Sigma}\right\}$$

If we now assume that $p_{\perp}^{o\pm} = \frac{1}{2} p_{\perp}^o$ and $p_{\shortparallel}^{o\pm} = \frac{1}{2} p_{\shortparallel}^o$ together with $g_o^{\pm}(y,x^2) = h(y,x^2)$, we obtain, using the identity $2x \frac{\partial h}{\partial x^2} = \frac{\partial h}{\partial x}$ and integrating by parts

(63)
$$0 < \left(\frac{B_o^2}{\mu_o} + 2p_{\perp}^o\right) - \frac{1}{3} \frac{p_{\perp}^{o2}}{p_{\shortparallel}^o} \eta$$

with η given by

(64)
$$\eta = \frac{\left(\int x^4 \frac{\partial h}{\partial x^2}\,dxdy\right)\left(\int y^2 \frac{\partial h}{\partial x^2}\,dxdy\right)}{\left(\int x^2 y \frac{\partial h}{\partial x^2}\,dxdy\right)^2} .$$

Lüst[10] using a macroscopic model, has derived eq. (63) with

$\eta = 1$. If $h(y,x^2)$ is monotone down in x^2 for every value of y, then, taking $h_1 = x^2|\partial h/\partial x^2|^{\frac{1}{2}}$ and $h_2 = y|\partial h/\partial x^2|^{\frac{1}{2}}$ and using Schwartz's inequality, we have $\eta > 1$ so that this class of functions, at any rate, for stability the guiding center theory imposes greater restrictions on the pressures than the fluid theory does. Agreement is obtained in limiting cases of functions belonging to this class. In particular, if we take the sequence of functions

$$h_j = e^{-j^2(y-y_o)^2} \frac{1}{1+(x^2)^j} \qquad y_o > 0$$

and define η_j as the value of η when h_j replaces h in (64), then we obtain

$$\lim_{j \longrightarrow \infty} \eta_j = 1$$

If we are not restricted to monotonically decreasing functions in x^2, the quantity η may be less than one or even negative (in which case eq. (63) is always satisfied), as the following example shows. We choose $h(y,x^2)$ such that

$$h(y,x^2) = H(y) \frac{x^2}{(1+x^2)^m} \quad .$$

Then

$$\frac{\partial h}{\partial x^2} = H(y) \left\{ \frac{1 - (m-1)x^2}{(1+x^2)^{m+1}} \right\} .$$

Referring to eq. (62), since $H(y) > 0$ [$h(y,x^2)$ is an expectation function], it suffices to show that

$$(65) \qquad \int_0^\infty \frac{1 - (m-1)x^2}{(1+x^2)^{m+1}}\, dx > 0 \ .$$

Letting $x = \tan\theta$ in eq. (65), we obtain

$$\int_0^{\pi/2} \left\{ m \cos^{2m}\theta - (m-1)\cos^{2(m-1)}\theta \right\} d\theta$$

$$= \frac{\pi}{4} \left\{ \frac{1\cdot 3\cdots(2m-3)}{2\cdot 4\cdots(2m-2)} \right\} > 0 \ .$$

In Figs. 5 and 6 we indicate the ranges of $p_{\shortparallel}^{o}$ and $p_{\perp}^{o}$ which are required for stability by eq. (63) and the condition $\nu^2 > 0$ when $\eta > 0$. We note that if $\eta > 6$, then for a fixed magnetic field strength, B_o, there exist maximal pressures $\overline{p_{\perp}^{o}}$ and $\overline{p_{\shortparallel}^{o}}$ such that if $p_{\perp}^{o} > \overline{p_{\perp}^{o}}$ or $p_{\shortparallel}^{o} > \overline{p_{\shortparallel}^{o}}$, stability is impossible. On the basis of such a prediction, it might be anticipated that for all the functions which we could admit to consideration (i.e., positive and vanishing at infinity in the appropriate μ-v half-plane) η must be less than six. The following counter-example shows otherwise:

$$h(y,x^2) = \frac{1}{(1+\mu)^m}\, e^{-x^2}$$

$$\eta = 6\,\frac{m-2}{m-3} > 6 \quad \text{if} \quad m \geq 4 \ .$$

Figures 7 and 8 are plots of the temperature ratios against $\beta = p_{\perp}^{o}/(B_{o}^{2}/\mu_{o})$.

Before taking leave of this criterion we note that all spherically symmetric, and consequently, all two-temperature Maxwellian distribution functions, have the value six for η.

As remarked above during the discussion of zeros of $f(z)$, it is not sufficient to insure exponential stability that the two conditions derived thus far should be satisfied. That is, $D(0;\lambda^2)$ may be negative, and there may still exist zeros of $D(z;\lambda^2)$ in the upper half plane. Therefore, we now turn our attention to the derivation of a condition which, together with the two necessary conditions for stability, will guarantee the stability of the equilibrium configuration. This condition, not surprisingly, will depend on microscopic properties of the plasma.

The sufficient condition we impose is that $D(z;\lambda^2)$ should never be real and non-negative for z real. As noted previously, together with $D(0;\lambda^2) > 0$, this implies that $D(z;\lambda^2)$ is never zero in the half plane.

Accordingly, suppose that $D(z;\lambda^2)$ is real for some z which is real and different than zero. Then for such a value of z, assuming $\lambda^2 \neq 0$ and

$$\int_{0}^{\infty} \mu\left(\frac{1}{\rho_o} f_{o,v}\right)_{\Delta} d\mu \Big|_{v=z} \neq 0 \ ,$$

we have, from Im $D(z;\lambda^2) = 0$, that[14]

$$(66)\qquad \mp_\mu\left(\frac{1}{\rho_o} f_{o,v}\right)_\Delta = \frac{1}{2} \frac{1}{\lambda^2 \frac{B_o^2}{\rho_o} |(\alpha\rho_o)^{\pm}|} \frac{1}{z^2 \int_o^\infty \mu\left(\frac{1}{\rho_o} f_{o,v}\right)_\Delta d\mu}$$

$$\times \left\{(\mathrm{Re}\ T(z))\left(-\lambda^2 \frac{B_o^2}{\rho_o} \int_o^\infty \mu^2 (f_{o,v})_\Sigma\, d\mu\right)\right.$$

$$\left. - z^2 \int_o^\infty \left(\frac{|\alpha|}{\rho_o} f_{o,v}\right)_\Sigma d\mu \left(z^2 - \nu^2 - \lambda^2\left(c^2 + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o} I_{\mu^2}(f_{o,v})_\Sigma\right)\right)\right\}$$

where $\mp_{\mu^k} = \text{P.V. } I_{\mu^k}$ for z real. Substituting this into the expression for Re $D(z;\lambda^2)$ when z is real. we obtain

$$(67)\qquad \mathrm{Re}\ D(z;\lambda^2) = (\mathrm{Re}\ T(z)\left\{z^2 - \nu^2 - \lambda^2\left(c^2 + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o} \mp_{\mu^2}(f_{o,v})_\Sigma\right)\right\}$$

$$- z^2\lambda^2 \frac{B_o^2}{\rho_o} \pi^2 \left\{\left(\int_o^\infty \left(\frac{|\alpha| f_{o,v}}{\rho_o}\right)_\Sigma d\mu \int_o^\infty \mu^2 (f_{o,v})_\Sigma\, d\mu - |\alpha\rho_o^{\pm}|\left(\int_o^\infty \mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2\right)\right\}$$

$$- \left\{(\mathrm{Re}\ T)\lambda^2 \frac{B_o^2}{\rho_o} \int_o^\infty (\mu^2 f_{o,v})_\Sigma d\mu + z^2\left(z^2 - \nu^2 - \lambda^2\left(c^2 + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o} \mp_{\mu^2}\left(f_{o,v}\right)_\Sigma\right)\right)\right.$$

$$\left.\times \int_o^\infty \left(\frac{|\alpha|}{\rho_o} f_{o,v}\right)_\Sigma d\mu\right\}^2 \frac{1}{4z^2\lambda^2 \frac{B_o^2}{\rho_o} |(\alpha\rho_o)^{\pm}|} \frac{1}{\left(\int_o^\infty \mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2}$$

If the first product appearing in eq. (67) is negative, then it suffices to have at $v = z$

$$(68) \qquad |\alpha^{\pm}\rho_o^{\pm}|\left(\int_o^\infty \mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2 < \int_o^\infty \left(|\alpha|\ \frac{f_{o,v}}{\rho_o}\right)_\Sigma d\mu \cdot \int_o^\infty \mu^2 (f_{o,v})_\Sigma\, d\mu$$

We now claim that this condition is a sufficient condition for stability when taken together with $\nu^2 > 0$ and eq. (62).

For suppose the first product is positive at the value of z in question. Then, upon expanding the square in the last term of (67), we obtain, after collecting like terms, a coefficient of the first product which is given by

$$(69) \qquad 1 - \frac{1}{2}\left.\frac{\int_o^\infty (\mu^2 f_{o,v})_\Sigma\, d\mu \int_o^\infty (\frac{|\alpha|}{\rho_o} f_{o,v})_\Sigma\, d\mu}{|\alpha^{\pm}\rho_o^{\pm}|\left(\int_o^\infty \left(\mu\ \frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2}\right|_{v=z}$$

which is less than

$$\frac{1}{2}\left.\frac{\int_o^\infty (\mu^2 f_{o,v})_\Sigma\, d\mu \int_o^\infty (\frac{|\alpha|}{\rho_o} f_{o,v})_\Sigma\, d\mu}{|\alpha^{\pm}\rho_o^{\pm}|\left(\int_o^\infty \left(\mu\ \frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2}\right|_{v=z}$$

if eq. (68) is satisfied. Therefore, under the hypothesis

of positivity of the first product and the validity of eq. (68), we have, at $v = z$,

$$(70)\quad \mathrm{Re}(D(z;\lambda^2)) < -z^2\lambda^2 \frac{B_o^2}{\rho_o}\left\{\int_o^\infty\left(\frac{|\alpha| f_{o,v}}{\rho_o}\right)_\Sigma d\mu \cdot \int_o^\infty (\mu^2 f_{o,v})_\Sigma \, d\mu\right.$$

$$\left. -|\alpha^{\pm}\rho_o^{\pm}|\left(\int_o^\infty \mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2\right\} - \frac{1}{4z^2\lambda^2 \frac{B_o^2}{\rho_o}|\alpha^{\pm}\rho_o^{\pm}|} \cdot \frac{1}{\left(\int_o^\infty \mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta d\mu\right)^2} \cdot$$

$$\cdot\left\{z^2\left\{z^2 - \nu^2 - \lambda^2\left(c^2 + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o} \mp \mu^2 (f_{o,v})_\Sigma\right)\right\}\int_o^\infty \left(|\alpha| \frac{f_{o,v}}{\rho_o}\right)_\Sigma d\mu\right.$$

$$\left. - \lambda^2 \frac{B_o^2}{\rho_o} (\mathrm{Re}\ T) \int_o^\infty (\mu^2 f_{o,v})_\Sigma \, d\mu\right\}^2 < 0 \ .$$

If $\int_o^\infty (\mu\, f_{o,v}/\rho_o)_\Delta \, d\mu = 0$, but our inequality, given by eq. (68), is satisfied, the remaining two terms have the same sign. From $\mathrm{Im}\ D(z;\lambda^2) = 0$ we obtain

$$(71)\qquad 0 = \lambda^2 \frac{B_o^2}{\rho_o} (\mathrm{Re}\ T(z))\left\{\int_o^\infty \mu^2 (f_{o,v})_\Sigma \, d\mu\right\}$$

$$+ z^2\left\{z^2 - \nu^2 - \lambda^2\left(c + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o} \mp \mu^2 (f_{o,v})_\Sigma\right)\right\}\int_o^\infty \left(\frac{|\alpha|}{\rho_o} f_{o,v}\right)_\Sigma d\mu$$

together with

$$(72)\qquad \mathrm{Re}\ D(z;\lambda^2) = \left\{(\mathrm{Re}\ T)\ z^2-\nu^2-\lambda^2\left(c^2 + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o}\ I_{\mu^2}(f_{o,v})_\Sigma\right)\right\}$$

$$- \lambda^2\ \frac{B_o^2}{\rho_o}\ z^2|\alpha^{\pm}\rho_o^{\pm}|\left\{\mp_\mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta\right\}^2 - \pi^2\lambda^2\ \frac{B_o^2}{\rho_o}\ z^2 \int_o^\infty d\mu\left(\frac{|\alpha|}{\rho_o}\ f_{o,v}\right)_\Sigma$$

$$\times \int_o^\infty \mu^2(f_{o,v})_\Sigma\ d\mu \quad .$$

However, since $\int_o^\infty \mu^2(f_{o,v})_\Sigma\ d\mu$ and $\int_o^\infty (\frac{|\alpha|}{\rho_o}\ f_{o,v})_\Sigma\ d\mu$ have the same sign, from eq. (71) we infer that Re T(z) and $\left\{z^2-\nu^2-\lambda^2(c^2 + \frac{p_\perp^o}{\rho_o} + \frac{B_o^2}{\rho_o}\ \mp_{\mu^2}(f_{o,v})_\Sigma)\right\}$ have opposing sign and, therefore, from eq. (72) we conclude that Re $D(z;\lambda^2) < 0$.

If $\lambda^2 = 0$, then on the real line $D(z;\lambda^2)$ becomes

$$(73)\qquad D(z;0) = (z^2-\nu^2)[1 + |\alpha|_\Sigma - z^2 \mp\left(\frac{|\alpha| f_{o,v}}{\rho_o}\right)_\Sigma$$

$$- \pi i z^2 \int_o^\infty \left(\frac{|\alpha| f_{o,v}}{\rho_o}\right)_\Sigma\ d\mu]$$

so that the only real values of z different than zero for which D(z;0) is real are given by roots of

$$(74)\qquad \int_o^\infty \left(\frac{|\alpha| f_{o,v}}{\rho_o}\right)_\Sigma\ d\mu\ \Bigg|_{v=z} = 0 \quad .$$

However, if eq. (74) is satisfied the inequality given by eq. (68) is not.

Therefore, $v^2 > 0$ together with the inequalities given by eq. (62) and (68) are sufficient to guarantee exponential stability.

We note that in the test prescribed in eq. (68) the kinetic properties of the equilibrium configuration are preserved, and as a result this sufficiency condition is very sensitive to the kinetic velocity profile of the distribution functions in μ-v space.

We now describe certain classes of functions which satisfy this criterion.

If $f_o^{\pm}(\mu,v^2)$ is monotone down in v^2 for every μ, then we claim that the inequality given by eq. (68) is always satisfied. This follows from an expansion of both sides of the inequality. On the left we have

$$|(\alpha\rho_o)^{\pm}|\left[\left(\int_o^{\infty}\mu\,\frac{f_{o,v}}{\rho_o}\,d\mu\right)^2\right]_{\Sigma} - 2|(\alpha\rho_o)^{\pm}|\int_o^{\infty}\mu\,\frac{f_{o,v}^{+}}{\rho_o^{+}}\,d\mu$$

(75)

$$\cdot\int_o^{\infty}\mu\,\frac{f_{o,v}^{-}}{\rho_o^{-}}\,d\mu < |(\alpha\rho_o)^{\pm}|\left[\left(\int_o^{\infty}\mu\,\frac{f_{o,v}}{\rho_o}\,d\mu\right)^2\right]_{\Sigma}$$

from the monotonicity of $f_o^{\pm}(\mu,v^2)$. On the right we obtain

$$(76) \qquad |(\alpha\rho_o)^{\pm}| \left[\int_o^\infty \frac{f_{o,v}}{\rho_o} \, d\mu \cdot \int_o^\infty \mu^2 \frac{f_{o,v}}{\rho_o} \, d\mu \right]_\Sigma$$

$$+ |\alpha^+|\rho_o^- \int_o^\infty \frac{f_{o,v}^+}{\rho_o^+} \, d\mu \int_o^\infty \mu^2 \frac{f_{o,v}^-}{\rho_o^-} \, d\mu + |\alpha^-|\rho_o^+ \int_o^\infty \frac{f_{o,v}^-}{\rho_o^-} \, d\mu \int_o^\infty \mu^2 \frac{f_{o,v}^+}{\rho_o^+} \, d$$

$$> |(\alpha\rho_o)^{\pm}| \left[\int_o^\infty \frac{f_{o,v}}{\rho_o} \, d\mu \cdot \int_o^\infty \mu^2 \frac{f_{o,v}}{\rho_o} \, d\mu \right]_\Sigma$$

using the monotonicity again. Using Schwartz's inequality with the functions $|f_{o,v}^{\pm} \rho_o^{\pm}|^{1/2}$ and $\mu|f_{o,v}^{\pm} \rho_o^{\pm}|^{1/2}$, we have that the right hand side of the inequality given by eq. (76) is greater than that given by eq. (75). This completes the proof.

Suppose now that $f_o^{\pm}(\mu,v^2)$ are monotone down in v^2 for $|v| < v_c^{\pm}(\mu)$ but are identically zero for $|v| > v_c^{\pm}(\mu)$ (i.e., $v_c^{\pm}(\mu)$ is a cutoff velocity depending upon μ). Then if there exists a maximal v_c for $v_c^{\pm}(\mu)$ which is finite, which we denote as $\overline{v_c}$, then we observe that at $z = \overline{v_c}$, the inequality given by eq. (68) fails and if $\mathrm{Re}(D(\overline{v_c},\lambda^2)) > 0$, a contour diagram, shown in Fig. 9 (or something qualitatively equivalent to it) might occur. We make the claim that this cannot happen. That is, functions which are monotone down in v^2, modulo a cutoff velocity, are also exponentially stable.

The proof is simple. Suppose these functions were not stable exponentially. Then in a small half neighborhood of $\overline{v_c}$ given by $v < \overline{v_c}$, we must have $\mathrm{Im}\, D(v,\lambda^2) > 0$. That is

to say, the contour diagram must produce a crossing of the real positive axis and not just touch it. However, this implies that in a neighborhood of this line in the complex upper half plane, the imaginary part of $D(z;\lambda^2)$ must also be positive, by the continuity of $D(z;\lambda^2)$. Consider, however, the sequences $g_n^{\pm}(\mu,v^2)$ defined by

$$(77)\qquad g_n^{\pm}(\mu,v^2) = \begin{cases} f_o^{\pm}(\mu,v_n^2)\exp\left\{-\frac{1}{2}\left|\dfrac{f_{o,v}^{\pm}(\mu,v_n^{\pm 2})}{v_n^{\pm} f_o^{\pm}(\mu,v_n^{\pm 2})}\right|(v^2-v_n^2)\right\} & \text{if } |v| > v_n^{\pm}(\mu) \\ f_o^{\pm}(\mu,v^2) & \text{if } |v| \le v_n^{\pm}(\mu) \end{cases}$$

where $v_n^{\pm}(\mu) < v_c^{\pm}(\mu)$ and $\lim_{n\to\infty} v_n^{\pm}(\mu) = v_c^{\pm}(\mu)$. We have that

$$(78)\qquad \lim_{n\to\infty} g_n^{\pm} = f_o^{\pm}$$

$$\lim_{n\to\infty} g_{n,v}^{\pm} = f_{o,v}^{\pm}$$

The $g_n^{\pm}$ are monotone down in v^2 for all values of μ and, therefore, satisfy our sufficient condition for stability. However, for every z belonging to the upper half plane, the mapping by $D(z;\lambda^2)$ depends continuously on the functions $f_{o,v}^{\pm}$.† Since the image of the boundary of the half plane, with $g_n^{\pm}$ replacing $f_o^{\pm}$, under $D(z;\lambda^2)$ does not cross the positive

† See appendix.

real axis, we have, for fixed z and sufficiently large n that near $\overline{v_c}$, the image of z is in the lower half plane. Thus, in the limit, the image must be in the lower half plane. This completes the proof.

We note, before leaving this subject, that the method of proof also yields the exponential stability of those functions which are only monotonically non-increasing in v^2 for every value of μ, provided they satisfy the other hypothesis we have imposed.

We conclude this section with a treatment of the problem excluded in the previous discussion of stability. That is, we assume the symmetry of the problem to be such that all spacial dependence of the solution is confined to the variables in the plane orthogonal to $\vec{B}_o$.

In this situation, our equations become

$$\frac{\partial}{\partial t}\left(f^{\pm} - \frac{f_o^{\pm}}{B_o} B_1\right) = 0 \tag{79}$$

$$\frac{\partial B_1}{\partial t} + \left(\frac{\partial}{\partial x_2} r_2 + \frac{\partial}{\partial x_3} r_3\right) = 0 \tag{80}$$

$$\frac{\partial}{\partial t} B_{2,3} = 0 \tag{81}$$

$$\frac{\partial r_{2,3}}{\partial t} + c^2 \frac{\partial B_1}{\partial x_{2,3}} + \frac{B_o^2}{\rho_o} \frac{\partial}{\partial x_{2,3}} \int \mu d\mu dv (f^+ + f^-) = 0. \tag{82}$$

Equations (79) and (81) are easily integrated and yield

$$B_{2,3}(t,x) = B_{2,3}(0,x)$$

$$f^{\pm}(t,x;\mu,v) = \frac{f^{\pm}_o}{B_o} B_1(t,x) + F^{\pm}(0,x;\mu,v) \tag{83}$$

where $F^{\pm}$ is a linear combination of the initial conditions and is time-independent.

Substituting the second of eq. (83) into eq. (82) we obtain

$$\frac{\partial r_{2,3}}{\partial t} + (c^2 + \frac{p^o_\perp}{\rho_o}) \frac{\partial B_1}{\partial x_{2,3}} = - \frac{\partial}{\partial x_{2,3}} G(0,\vec{x}) \tag{84}$$

where

$$G(0,x) = \frac{B_o^2}{\rho_o} \int \mu d\mu dv [F^+(\mu,v) + F^-(\mu,v)] \ .$$

Assuming that our functions have two continuous derivatives, we may eliminate r_2 and r_3 to obtain a wave equation for B_1 with wave speed $\left\{\frac{1}{\rho_o} (B_o^2/\mu_o + 2p^o_\perp\right\}^{1/2}$

$$\frac{\partial^2 B_1}{\partial t^2} - (c^2 + \frac{p^o_\perp}{\rho_o})(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2})B_1 = (\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}) G(0,\vec{x}). \tag{85}$$

The solution to this equation and its properties are well known.[17] It suffices for us to note that the wave speed is real and that no additional criteria for exponential stability are introduced in this singular case.

The appearance of this wave speed will be accounted for in the next section.

V. Some Asymptotic Results Pertaining to Wave Propagation

We recall that in the treatment of propagation orthogonal to $\bar{B}_o$, it was observed that signals propagated with speed $\left\{\frac{1}{\rho_o}\left(B_o^2/\mu_o + 2p_\perp^o\right)\right\}^{1/2}$. Since a cursory inspection of $D(z;\lambda^2)$ does not predict such a wave speed, we now will show that the result is obtained from this function by an asymptotic expansion of λ^2.

Defining the wave speed $\sigma = z\cos\theta = \omega/|k|$, we note that non-zero σ and $\theta \to \pi/2$ implies that $|z| \to \infty$. Thus, we allow λ^2 to become large ($\theta \to \pi/2$) and look for non-zero wave speeds which yield zeros of $D(z;\lambda^2)$. Since we have just seen that the existence of such a propagation speed implies $|z| \to \infty$, we may employ the asymptotic expansion† of $D(z;\lambda^2)$ for large z and obtain,

$$D(z;\lambda^2) \sim -\frac{\sin^2\theta}{\sigma^2}\,\frac{(|\alpha|\rho_o)^{\pm}}{\rho_o}\left[\left(\frac{p_\perp^o}{\rho_o}\right)_\Delta\right]^2$$
$$+\left\{1 - \frac{3\,\frac{|\alpha|p_\parallel^o}{\rho_o}\,\Sigma}{\sigma^2}\cos^2\theta\right\}\left\{\frac{\sigma^2}{\cos^2\theta} - \left[\nu^2 + (c^2 + \frac{p_\perp^o}{\rho_o})\tan^2\theta\right.\right.$$
$$\left.\left. + \int \mu d\mu dv\, f_\Sigma^o\left(\frac{B_o^2}{\rho_o}\,\frac{\sin^2\theta}{\sigma^2}\right)\right]\right\}$$

Factoring $1/\cos^2\theta$ and retaining only the dominant part of the expression we see that

$$D(z;\lambda^2) \sim \frac{1}{\cos^2\theta}\left\{\sigma^2 - (c^2 + \frac{p_\perp^o}{\rho_o})\sin^2\theta\right\}$$

† See appendix.

and this is the desired result.

Some further asymptotic results may be obtained if it is assumed that the ratio of ion and electron thermal speeds is much smaller than unity (i.e., $v_{th}^{+}/v_{th}^{-} \ll 1$) and we investigate the existence of zeros of $D(z;\lambda^2)$ which are large with respect to v_{th}^{+} but small when compared with v_{th}^{-}.

We will assume that $f_o^{\pm}(\mu,v)$ are of the form

$$f_o^{\pm}(\mu,v) = \frac{B_o \rho_o^{\pm}}{(v_{th}^{\pm})^3}\, g^{\pm}\left(\frac{B_o \mu}{(v_{th}^{\pm})^2}\,,\ \frac{\frac{1}{2}v^2}{(v_{th}^{\pm})^2}\right)$$

and define the quantities $\xi^{\pm}$ as

$$\xi^{\pm} = \frac{z}{v_{th}^{\pm}}$$

thereby being led to solutions with $|\xi^{+}| \gg 1$ and $|\xi^{-}| \ll 1$.

Defining the dimensionless variables

$$y = \frac{B_o \mu}{v_{th}^2} \quad ; \qquad x = \frac{v}{v_{th}}$$

we obtain

$$\int \mu^k \frac{f_{o,v}^{\pm}}{v-z}\, dv d\mu = \frac{\rho_o}{B_o^k}\,(v_{th}^2)^{k-1} \int \frac{y^k g_x^{\pm}(y,\ \frac{1}{2}x^2)}{x-\xi^{\pm}}\, dy dx \ .$$

Therefore, in terms of these dimensionless variables, we find

$$(86)\qquad D(z;\lambda^2) = \left\{(1+|\alpha|_\Sigma) - z^2\left(\frac{|\alpha|}{v_{th}^2}\int \frac{g_x}{x-\xi}\,dydx\right)_\Sigma\right\}\cdot$$

$$\cdot\left\{z^2-(\nu^2+\lambda^2(c^2+\frac{p_\perp^o}{\rho_o})) - \lambda^2(\rho_o v_{th}^2\int \frac{y^2 g_x}{x-\xi}\,dydx)_\Sigma\right\}$$

$$- z^2\frac{\lambda^2}{\rho_o}(|\alpha|\rho_o)^{\pm}\left\{\left(\int y\,\frac{g_x}{x-\xi}\,dydx\right)_\Delta\right\}^2$$

We now make use of the assumptions $|\xi^+| >> 1$ and $|\xi^-| << 1$ to obtain the following expansions of the integrals in eq. (86):

$$(87)\qquad \int \frac{y^k g_x^+}{x-\xi^+}\,dxdy \sim \frac{1}{\xi_+^2}\left\{\int y^k g^+ dxdy + \frac{3}{\xi_+^2}\int x^2 y^k g\,dxdy + O(\frac{1}{\xi_+^4})\right.$$

$$\int \frac{y^k g_x^-}{x-\xi^-}\,dxdy \sim \int y^k g_{x^2}^-\,dxdy + \pi i\xi^-\int y^k g_{xx}(y,0)dy$$

$$+ O(\xi_-^2)$$

where we have assumed, for convenience, two derivatives for g^-.

Substituting the expansions of eq. (87) into eq. (86) we obtain the asymptotic result

$$(88)\qquad D(z;\lambda^2) \sim v_{th}^{+2}(1+|\alpha|_-)\left\{\xi_+^2 - \frac{\nu^2+\lambda^2(c^2+p_\perp^o/\rho_o)}{v_{th}^{+2}}\right.$$

$$\left. - \frac{\lambda^2}{\rho_o}\rho_o^-\left(\frac{v_{th}^-}{v_{th}^+}\right)^2\int y^2 g_{x^2}^-\,dxdy\right\} - \xi_+^2\frac{\lambda^2}{\rho_o}(|\alpha|\rho_o)^{\pm}(|yg_{x^2}^-dxdy)^2 v$$

In eq. (88), only the dominant terms have been retained. It has been assumed that $(\rho_o v_{th}^2)^{\pm}$ are of the same order of magnitude, and, in addition, integrals of ion and electron functions which appear are of the order of unity. The first of these assumptions, together with charge neutrality and the hypothesis $v_{th}^+/v_{th}^- << 1$ allows us to assume that $m^-/m^+ << 1$ so that we may take $|\alpha|^+ = 0$ and $|\alpha|^- = 1$, together with $\rho_o = \rho_o^+$ for our purposes.

Writing $\xi_+ = \xi \cos\theta$, we find that to lowest order the zeros of $D(z;\lambda^2)$ are given by

$$\xi^2 = \frac{\frac{1}{v_{th}^{+2}}\left\{\nu^2\cos^2\theta + (c^2 + \frac{p_\perp^o}{\rho_o})\sin^2\theta\right\}}{\left\{1 - \frac{\lambda^2}{2}\frac{\rho_o^-}{\rho_o^+}\left(\int y g_{x^2}^- \, dydx\right)^2\right\}}$$

$$+ \frac{(\sqrt{\rho_o^-/\rho_o}\; v_{th}^-/v_{th}^+)^2 \int y^2 g_{x^2}^- \, dxdy \sin^2\theta}{\left\{1 - \frac{\lambda^2}{2}\frac{\rho_o^-}{\rho_o^+}\left(\int y g_{x^2}^- \, dydx\right)^2\right\}}$$

If $\lambda^2 \rho_o^-/\rho_o^+ << 1$ (i.e., θ is not near $\pi/2$), the denominators may be approximated by unity, so that in order that $|\xi_+| >> 1$ we must have

$$\frac{\frac{1}{\rho_o}\left\{(B_o^2/\mu_o + p_\perp^o) + p_\perp^o \sin^2\theta - p_{\shortparallel}^o \cos^2\theta\right\}}{v_{th}^{+2}} >> 1$$

Assuming that $\rho_o^+ v_{th}^{+2}$ is of the order of the fluid pressures, another way of expressing the result is that for the waves to exist it is necessary that the ratio of the magnetic to the fluid pressures be much larger than unity. This, of course, corresponds to small β in Figs. 7 and 8.

VI. Fundamental Solution for the One-Dimensional Problem

We proceed now to find a fundamental solution for the one-dimensional problem associated with equations (11)-(13). It is assumed that the independent space variable, s, lies on an axis in the x-y plane making an angle θ with the magnetic field line. Under these hypotheses, the equations become

$$(89)\qquad 0 = \left(\frac{\partial}{\partial t} + v\cos\theta\,\frac{\partial}{\partial s}\right)f^{\pm} + \left\{\left([\eta]\alpha^{\pm} - \mu\right)f^{\pm}_{o,v} - v\,\frac{f^{\pm}_o}{B_o}\right\}\cos\theta\,\frac{\partial}{\partial s}B_1 + \frac{f^{\pm}_o}{B_o}\sin\theta\,\frac{\partial}{\partial s}r_2 + \alpha^{\pm}f^{\pm}_{o,v}\cos\theta\,\frac{\partial}{\partial s}\,v^2$$

$$(90)\qquad 0 = \frac{\partial}{\partial t}B_1 + \sin\theta\,\frac{\partial}{\partial s}r_2$$

$$0 = \frac{\partial}{\partial t}B_{2,3} - \cos\theta\,\frac{\partial}{\partial s}r_{2,3}$$

$$0 = \frac{\partial r_2}{\partial t} - \nu^2 \cos\theta \frac{\partial}{\partial s} B_2 + c^2 \sin\theta \frac{\partial}{\partial s} B_1 + \frac{B_o^2}{\rho_o} \sin\theta \frac{\partial}{\partial s} \langle \mu \rangle$$

(90)

$$0 = \frac{\partial r_3}{\partial t} - \nu^2 \cos\theta \frac{\partial}{\partial s} B_3 \ .$$

We define, as before, the vector

(91) $$\vec{u}(t) = (f^+, f^-, B_1, B_2, r_2, B_3, r_3)$$

with initial data

(92) $$\vec{u}(0) = (g^+, g^-, b_1, b_2, R_2, b_3, R_3)\delta(s)$$

Since the problem for $\theta = \pi/2$ has been solved explicitly in a preceding section, we assume $\theta \neq \pi/2$ and divide our equations by $\cos\theta$. Introducing the new variable $t \cos\theta = \tau$, we Fourier transform the equations in s and take a Laplace transform in τ to obtain

(93) $$(p + ik\nu)\hat{f}^{\pm} + \left\{([\eta]\alpha^{\pm} - \mu) f^{\pm}_{o,v}\right\} ik\hat{B}_1 + ik\lambda \frac{f^{\pm}_o}{B_o} \hat{r}_2 + \alpha^{\pm} f_{o,v}\, ik\langle \hat{v}^2 \rangle = g^{\pm}(\mu, v)$$

(94)

$$p\hat{B}_1 + \lambda ik\hat{r}_2 = b_1$$

$$p\hat{B}_{2,3} - ik\hat{r}_{2,3} = b_{2,3}$$

$$\hat{pr}_2 - ik\nu^2\hat{B}_2 + ikc^2\lambda\hat{B}_1 + ik\lambda\frac{B_o^2}{\rho_o}\langle\hat{\mu}\rangle = R_2$$

(95)

$$\hat{pr}_3 - ik\nu^2\hat{B}_3 = R_3$$

where, as before, p is the Laplace transform variable, k is the Fourier transform variable, and $\lambda = \tan\theta$.

We observe that eq. (93)-(95) are, after defining $p/ik = -z$, exactly eq. (43)-(54) with a slightly modified right hand side. We readily obtain, therefore, eq. (46) with a modified right hand side given by

(96) $$\frac{1}{ik}\left(\int \mu\,\frac{g^+ + g^-}{v-z}\,d\mu dv,\ \int \nu^2\,\frac{g^+/\rho_o^+ - g^-/\rho_o^-}{v-z}\,d\mu dv,\ b_1,\ b_2,\ R_2,\ b_3,\ R_3\right)$$

Writing the matrix A as

(97) $$A = \begin{pmatrix} M & & 0 \\ & -z & -1 \\ 0 & & \\ & -\nu^2 & -z \end{pmatrix}$$

as in eq. (48'), we have

(98) $$A^{-1} = \begin{pmatrix} M^{-1} & & 0 \\ & -\frac{z}{z^2-\nu^2} & \frac{1}{z^2-\nu^2} \\ 0 & & \\ & \frac{\nu^2}{z^2-\nu^2} & -\frac{z}{z^2-\nu^2} \end{pmatrix}$$

(99) $$M^{-1} = -\frac{1}{zD} \times$$

$$\begin{bmatrix}
-z(z^2-\omega_\lambda^2)T & -z(z^2-\omega_\lambda^2)I_\mu(|\alpha|f_{o,v})_\Delta & -(z^2-\nu^2)\{I\} & \nu^2\lambda\{I\} & -z\lambda\{I\} \\
z\frac{\lambda^2 B_o^2}{\rho_o}\{II\} & -z(z^2-\omega_\lambda^2-\lambda^2\frac{B_o^2}{\rho_o}\{II\}) & -(z^2-\nu^2)\{II\} & \nu^2\lambda\{II\} & -z\lambda\{II\} \\
-z\frac{\lambda^2 B_o^2}{\rho_o}T & -z\lambda^2\frac{B_o^2}{\rho_o}I_\mu(|\alpha|f_{o,v})_\Delta & (z^2-\nu^2)T & -\nu^2\lambda T & -z\lambda T \\
z\lambda\frac{B_o^2}{\rho_o}T & z\lambda\frac{B_o^2}{\rho_o}I_\mu(|\alpha|f_{o,v})_\Delta & \frac{D-(z^2-\nu^2)T}{\lambda} & D+\nu^2 T & -zT \\
-z^2\lambda\frac{B_o^2}{\rho_o}T & -z^2\lambda\frac{B_o^2}{\rho_o}I_\mu(|\alpha|f_{o,v})_\Delta & -\frac{z}{\lambda}D-(z^2-\nu^2)T & -z\nu^2 T & z^2 T
\end{bmatrix}$$

where

$$\{I\} = -\left\{\left(\frac{p_\perp^o}{B_o^2} + I_{\mu^2}(f_{o,v})_\Sigma\right)T + (|\alpha|\rho_o)^{\pm}z^2[I_\mu(\frac{f_{o,v}}{\rho_o})_\Delta]^2\right\}$$

$$\{II\} = \left\{[\eta]T - z^2 I_\mu(\frac{f_{o,v}}{\rho_o})_\Delta\right\}$$

$$\omega_\lambda^2 = \nu^2 + \lambda^2 c^2$$

We shall first treat propagation parallel to the magnetic field vector, $\overline{B}_o$. In this situation $\lambda = 0$ and $D(z;0) = (z^2-\nu^2)T$. As a result, M^{-1} is given in the relatively simple form

$$
(100)\qquad M^{-1} = \begin{pmatrix}
1 & \dfrac{(\alpha\rho_0)^{\frac{1}{2}}[I_\mu(\frac{f_{0,\nu}}{\rho_0})_{\angle}]}{T} & \dfrac{\{I\}}{z} & 0 & 0 \\
0 & \dfrac{1}{T} & \dfrac{\{II\}}{zT} & 0 & 0 \\
0 & 0 & -\dfrac{1}{z} & 0 & 0 \\
0 & 0 & 0 & \dfrac{-z}{z^2-\nu^2} & \dfrac{1}{z^2-\nu^2} \\
0 & 0 & 0 & \dfrac{\nu^2}{z^2-\nu^2} & \dfrac{-z}{z^2-\nu^2}
\end{pmatrix}
$$

If $\hat{q}$ is one of our transformed dependent variables, then $q(\tau,s)$ is given by

$$
(101)\qquad q(\tau,s) = \mathrm{Re}\,\frac{1}{2\pi^2}\int_0^\infty \kappa d\kappa \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{i\kappa(s-z\tau)}\,\hat{q}(z,\kappa)dz
$$

where σ is chosen sufficiently large so that all residues of $\hat{q}(z;\kappa)$ have imaginary part less than σ. We shall assume that T has no zeros in the upper half plane $\mathrm{Im}\, z \geq 0$.

We first observe that for all finite values of λ, we have

$$
(102)\qquad B_3(\tau,s) = \mathrm{Re}\,\frac{1}{2\pi^2 i}\int_0^\infty d\kappa \int_{-\infty+i\sigma}^{\infty+i\sigma} e^{i\kappa(s-z\tau)}\,\frac{dz}{z^2-\nu^2}\,(R_3 - zb_3)
$$

$$
r_3(\tau,s) = \mathrm{Re}\,\frac{1}{2\pi^2 i}\int_0^\infty d\kappa \int_{-\infty+i\sigma}^{\infty+i\sigma} e^{i\kappa(s-z\tau)}\,\frac{dz}{z^2-\nu^2}\,(\nu^2 b_3 - zR_3)
$$

Using the definition of the δ-function[18]

$$\delta(x) = \mathrm{Re}\,\frac{1}{\pi}\int_0^{\infty} e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk \tag{103}$$

we immediately obtain

$$\begin{aligned} B_3(\tau,s) &= \tfrac{1}{2}\,b_3[\delta(s+v\tau) + \delta(s-v\tau)] \\ &\quad - \tfrac{1}{2}\,\frac{R_3}{v}\,[\delta(s+v\tau) - \delta(s-v\tau)] \\ r_3(\tau,s) &= \tfrac{1}{2}\,R_3[\delta(s+v\tau) + \delta(s-v\tau)] \\ &\quad - \tfrac{1}{2}\,vb_3[\delta(s+v\tau) - \delta(s-v\tau)] \end{aligned} \tag{104}$$

which are simply fundamental solutions to the wave equation with wave speed v. Similarly, we obtain identical results for $B_2(\tau,s)$ and $r_2(\tau,s)$ by simply changing the subscripts in eq. (104) from 3 to 2 for this special angle of propagation.

For $B_1(\tau,s)$ we easily obtain from eq. (90)

$$B_1(\tau,s) = b_1\delta(s)\ . \tag{105}$$

The computations for $\langle\mu\rangle$ and $\langle v^2\rangle$ are more subtle. We have

$$(106)\qquad \langle \mu \rangle = \mathrm{Re}\ \frac{1}{2\pi i} \int_0^{\infty} dk \int_{-\infty+i\sigma}^{\infty+i\sigma} dz\ e^{ik(s-z\tau)} \left\{ \int \mu\, \frac{g^+ + g^-}{v-z}\, d\mu dv \right.$$

$$\left. + \frac{(|\alpha|\rho_o)^{\pm}}{T}\, I_\mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta \cdot \left\{ \frac{v^2(g^+/\rho_o^+ - g^-/\rho_o^-)}{v - z} + \frac{b_1}{z}\{I\} \right\} \right\}$$

Here, we are unable, in general, to continue the integrand analytically into the lower z half plane. We note, however, that the k integration may be continued bo the negative k-axis without contributing to the integral,[19] and, therefore, the range of the integration may be taken over the entire real k-axis. Utilizing the fact that T has no zeros in Im $z \geq 0$, we displace the contour of integration in the z-plane to the real line and obtain, changing the order of integration

$$(107)\qquad \langle \mu(\tau,s) \rangle = \frac{1}{\tau} \left\{ \int \mu g_\Sigma(\mu, \tfrac{s}{\tau}) d\mu + \frac{(|\alpha|\rho_o)^{\pm}}{\pi} \right.$$

$$\mathrm{Im} \left[\frac{I_\mu\left(\frac{f_{o,v}}{\rho_o}\right)_\Delta I v^2 \left(\frac{g}{\rho_o}\right)_\Delta}{T} \right]_{z=\frac{s}{\tau}} + b_1 \left[\left\{ \frac{p_\perp^o}{B_o^2} + I_{\mu^2} \frac{(f_{o,v})_\Sigma}{v} \right\} \delta(s) \right.$$

$$\left. \left. + \frac{\mathrm{P.V.}}{\pi} \left(\frac{1}{s}\right) \mathrm{Im} \{ I \}\Big|_{\frac{s}{\tau}} \right] \right\}$$

Similarly,

$$(108)\qquad \langle v^2(\tau,s) \rangle = \frac{1}{\tau} \left\{ \frac{(|\alpha|\rho_o)^{\pm}}{\pi}\, \mathrm{Im} \left[\frac{I(v^2(g/\rho_o)_\Delta}{T} \right]_{z=s/\tau} \right\}$$

$$- b_1 \left[[\eta]\delta(s) + \frac{1}{\pi\tau} \left(\frac{s}{\tau}\right) \mathrm{Im} \left[\frac{I_\mu(f_{o,v}/\rho_o)_\Delta}{T} \right]_{z=s/\tau} \right]$$

We note that in eq. (107) if we assume that $f_o(\mu,v)$ is even in v with a partial derivative in v which goes to zero as v^p ($p \geq 1$), then the principle value notation is not necessary since $Im\{I\}$ consists of a sum of terms, all of which contain a factor of the form $\int_o^\infty \mu^k d\mu \, f^{\pm}_{o,v}(\mu,v)\Big|_{v=s/t}$ ($k = 0,1,2$).

The behavior of $\vec{B}$ and $\vec{r}$ for the various ranges of s and τ are easily found from the δ-functions. Specifically, $B_{2,3}$ and $r_{2,3}$ are zero except on the wave fronts $s/\tau = \pm v$, along which an undamped disturbance is seen to propagate. B_1 is, of course, constant in time.

Inspection of eq. (107) and (108) shows that, in general, on every wave front s/τ = constant, there is a signal which is observed to damp in time like τ^{-1}. Fixing the time and considering the behavior of the signal as experienced at points in physical space at large distances from the initial disturbance, $\langle \mu \rangle$ and $\langle v^2 \rangle$ are seen to decay in distance like $\int_o^\infty \mu^k (g_{f_{o,v}})(\mu, s/\tau)d\mu$ roughly speaking, so that the effect of this disturbance is to be attributed to the fast particles, as was to be expected. On the other hand, if we consider the effect of the signal at any single position after a large time, we observe a much slower damping, with asymptotic behavior in time dominated by the slow particles. In fact

$$\langle \mu \rangle \sim \frac{1}{\tau} \int_o^\infty \mu g_\Sigma(\mu,0)d\mu \qquad (109)$$

(109)
$$\langle v^2\rangle \sim \frac{-1}{(1+|\alpha|_\Sigma)}\,\frac{s}{\tau^2}\,\Big(b_1\int_0^\infty \mu\left(\frac{f_{s/\tau}(\mu,s/\tau)}{\rho_o}\right)_\Delta d\mu$$
$$-\,(|\alpha|\rho_o)^{\pm}\int_0^\infty\left(\frac{g(\mu,0)}{\rho_o}\right)_\Delta d\mu\Big)$$

Assuming $\hat{B}_1$ and $\langle\hat{v}^2\rangle$ are known functions of z, we easily solve for $\hat{f}^{\pm}$ in eq. (93) to obtain

(110)
$$\hat{f}^{\pm} = \frac{1}{v-z}\left\{\frac{g^{\pm}}{ik} - F^{\pm}(\mu,v)\hat{B}_1 - \alpha^{\pm}f^{\pm}_{o,v}\langle\hat{v}^2\rangle\right\}$$

with
$$F^{\pm}(\mu,v) = \left\{([\eta]\alpha^{\pm} - \mu)f^{\pm}_{o,v} - v\,\frac{f^{\pm}_o}{B_o}\right\}.$$

Using the previous methods, and accounting for the singularity on the real line at z = v, we find

(111)
$$f^{\pm} = \left\{g^{\pm} - \mathrm{Re}(F^{\pm}B^*(v) + \alpha^{\pm}f^{\pm}_{o,v}\langle v^2(v)\rangle^*)\right\}\delta(s-vt)$$
$$+\,\frac{1}{\pi}\,\mathrm{P.V.}\,\frac{1}{s-vt}\,\mathrm{Im}(F^{\pm}B^*(\tfrac{s}{t}) + \alpha^{\pm}f^{\pm}_{o,v}\langle v^2(\tfrac{s}{t})\rangle^*)$$

where $q^*(z) = ik\hat{q}(z,k)$.

In the same way we obtain the representation for $\lambda^2 > 0$. The major distinction between the two cases is that for $\lambda^2 > 0$, the effect of a disturbance in B_2 and r_2 is felt by $f^{\pm}$ and B_1 and vice-versa. That is to say, B_2 and r_2 remain coupled to these variables when the wave vector is

not parallel to $\vec{B}_0$. Also, if $D(z;\lambda^2)$ does not have real zeros in the z variable, undamped wave propagation is only observed in the solutions for r_3 and B_3, which are, of course, still independent of disturbances in the other quantities.

Rather than writing down the solutions, which are very lengthy, we present the recipe which is valid for all of the quantities $\langle\mu\rangle$, $\langle v^2\rangle$, B_1, B_2, and r_2. One need only identify the appropriate elements of the matrix in eq. (99).

Let $q(t,s)$ be one of the five variables mentioned above. If $D(z;\lambda^2)$ has no zeros for real z, then $\hat{q}(z;k)$ is of the form

$$(112)\qquad \hat{q}(z,k) = \frac{1}{ik}\left\{A_1(z,\lambda)\int \mu\,\frac{g_\Sigma}{v-z}\,d\mu dv + A_2(z,\lambda)\int \frac{v^2(g/\rho_o)_\Delta}{v-z}\,d\mu dv + \frac{A_3(z;\lambda)}{z}\,b_1 + \frac{A_4(z,\lambda)}{z}\,b_2 + A_5(z,\lambda)R_2\right\}$$

where $A_i(z,\lambda)$ is analytic in z for Im $z > 0$ and bounded for Im $z \geq 0$. When $q(t;s) = r_2(t,s)$, A_3 and A_4 contain a factor z, so that the exhibited singularity at the origin does not, in this special case, exist.

Thus, we obtain

$$(113)\qquad q(\tau,s) = \frac{1}{\pi\tau}\,\mathrm{Im}\left\{A_1(z,\lambda)I_\mu(g_\Sigma) + A_2(z,\lambda)Iv^2(g/\rho_o)_\Delta + \mathrm{P.V.}\left[\frac{A_3(z,\lambda)b_1 + A_4(z,\lambda)b_2}{z}\right] + A_5(z,\lambda)R_2\right\}_{z=s/\tau} + \mathrm{Re}\left\{A_3(0,\lambda)b_1 + A_4(0,\lambda)b_2\right\}\delta(s).$$

The representations for f^{+-} are obtained exactly as before, with the trivial exception of the addition of a term due to the fact that $r_2(t,s)$ does not separate with $B_2(t,s)$ from the system.

Appendix

We will be concerned, in this section, with properties of functionals of the form

$$(A.1) \qquad I_{\mu^k} = \int_{-\infty}^{+\infty} v^{2n} \left(\int_0^{\infty} \mu^k f_{o,v} \, d\mu \right) \frac{dv}{v-z}$$

when z is a complex number in the half plane Im $z > 0$, $n = 0,1$ and $k = 0,1,2$.

If $n = 1$, we may reduce the above integral to the case $n = 0$ by adding and subtracting z twice. Thus,

$$(A.2) \qquad \int_{-\infty}^{+\infty} v^2 \left(\int_0^{\infty} \mu^k f_{o,v} \, d\mu \right) \frac{dv}{v-z} = \int_{-\infty}^{+\infty} \frac{v(v-z)+vz}{v-z} \left(\int_0^{\infty} \mu^k f_{o,v} \, d\mu \right) dv$$

$$= \int_{-\infty}^{+\infty} v\,dv \int_0^{\infty} \mu^k f_{o,v} \, d\mu + z \left(\int_{-\infty}^{+\infty} dv \int_0^{\infty} \mu^k f_{o,v} \, d\mu \right) + z^2 \int_{-\infty}^{+\infty} \frac{dv}{v-z} \int_0^{\infty} \mu^k f_{o,v} \, dv$$

After an integration by parts, we obtain

$$(A.3) \qquad \int_{-\infty}^{+\infty} v^2 \frac{dv}{v-z} \int_0^{\infty} \mu^k f_{o,v} d\mu = z^2 \int_{-\infty}^{+\infty} \frac{dv}{v-z} \int_0^{\infty} \mu^k f_{o,v} \, d\mu$$

$$- \int_{-\infty}^{+\infty} dv \int_0^{\infty} \mu^k d\mu \, f_o$$

When $k = 0$ or 1, the last term is a mass density or a

perpendicular pressure divided by the magnetic field.

Expanding the denominator of (A.1) in inverse powers of z, we have

$$(A.4)\qquad \int_{-\infty}^{+\infty} v^{2n}\Big(\int_{0}^{\infty} \mu^{k} f_{o,v}\, dv\Big)\,\frac{dv}{v-z}$$

$$= -\frac{1}{z}\int_{-\infty}^{+\infty} v^{2n} dv \int_{0}^{\infty} \mu^{k} f_{o,v}\, dv\Big(1 + \frac{v}{z} + \ldots + \Big(\frac{v}{z}\Big)^{p-1} + \frac{(v/z)^{p}}{1 - v/z}\Big)$$

$$= \frac{1}{z}\int_{0}^{\infty} \mu^{k} d\mu \int_{-\infty}^{+\infty} f_{o} dv\Big(2n\, v^{2n-1} + (2n+1)\frac{v^{2n}}{z} + \ldots + \frac{1}{z^{p}}\,\frac{\partial}{\partial v}\Big(\frac{v^{p+2n}}{1 - v/z}\Big)$$

so that for large values of z, we obtain

$$(A.5)\qquad \int_{-\infty}^{+\infty} v^{2n} \int_{0}^{\infty} \mu^{k} f_{o,v} dv\,\frac{dv}{v-z} \sim \frac{1}{z}\int_{0}^{\infty} \mu^{k} d\mu \int_{-\infty}^{+\infty} 2n\, v^{2n-1}\, f_{o} dv$$

$$+ \frac{1}{z^{2}}\int_{0}^{\infty} \mu^{k} d\mu \int_{-\infty}^{+\infty} (2n+1) v^{2n}\, f_{o} dv\ .$$

In particular, if $f_{o}(\mu,v)$ is even in v, the first term in the expansion vanishes, so that

$$(A.6)\qquad \int_{-\infty}^{+\infty} v^{2n} \int_{0}^{\infty} \mu^{k} f_{o,v}\,\frac{dv}{v-z} \sim \frac{2n+1}{z^{2}} \int_{0}^{\infty} \mu^{k} d\mu \int_{-\infty}^{+\infty} v^{2n}\, f_{o} dv\ .$$

We now show that if $\Big\{\int_{0}^{\infty} \mu^{k} f_{o,v}^{(n)}\, d\mu\Big\}$ is a convergent sequence in L_{p} $(p \geq 1)$ then the sequence of functionals $I_{\mu^{k}}^{(m)}$ obtained by replacing $f_{o,v}^{(m)}$ in (A.1) converges. In view of (A.3) it

suffices to prove this for n = 0. Let z = x+iy. Then

$$(A.7)\qquad |I_{\mu^k}^{(m)} - I_{\mu^k}^{(m+\ell)}| = \left|\int_{-\infty}^{+\infty} \frac{dv}{v-z} \int_0^\infty d\mu(\mu^k f_{o,v}^{(m)} - f_{o,v}^{(m+\ell)})\right|$$

$$\leq \left\| \int_0^\infty \mu^k d\mu (f_{o,v}^{(m+\ell)} - f_{o,v}^{(m)}) \right\|_p \cdot \left(\left\| \frac{v-x}{(v-x)^2+y^2} \right\|_{p/p-1} \right.$$

$$\left. + \left\| \frac{y}{(v-x)^2+y^2} \right\|_{p/p-1} \right)$$

Since $\left\{ \int_0^\infty \mu^k d\mu\ f_{o,v}^{(m)} \right\}$ is Cauchy in L_p, this completes the proof.

If we assume, once more, that $f_o(\mu,v)$ is even in v, then Re I_{μ^k} is even and Im I_{μ^k} is odd in Re z. Again, referring to (A.3), we need only prove this for n = 0.

$$(A.8)\qquad \int_{-\infty}^{+\infty} \frac{dv}{v-z} \int_0^\infty \mu^k d\mu\ f_{o,v} = \int_{-\infty}^{+\infty} \frac{dv}{(v-z)^2} \int_0^\infty \mu^k d\mu\ f_o$$

$$= \int_{-\infty}^{+\infty} \frac{dv}{|v-z|^4} \left\{ (v-x)^2 + 2iy(v-x) - y^2 \right\} \cdot \int_0^\infty \mu^k d\mu\ f_o \quad .$$

Suppose in (A.8) x is replaced by -x. Changing variables ($v = -v'$), we have $|v-z| = |v'+\bar{z}|$, $(v-x)$ simply changes sign, so that, replacing the dummy variable v' by v'', the real part of the integral is unchanged but the imaginary part is negated.

We now show that $f_o = f_o(\mu B_o + \frac{1}{2} v^2)$. Then $p_\perp^o = p_\parallel^o$. From the definitions, we have

$$p_\parallel^o = 2 \int_o^\infty v^2 dv \int_o^\infty f_o(\mu B_o + \tfrac{1}{2} v^2) d\mu$$

$$p_\perp^o = 2 \int_o^\infty dv \int_o^\infty \mu B_o f_o(\mu B_o + \tfrac{1}{2} v^2) d\mu$$

Letting $\mu B_o + \frac{1}{2} v^2 = \xi$ and changing the orders of integration, we obtain

$$p_\parallel^o = \frac{2}{B_o} \int_o^\infty g(\xi) d\xi \int_o^{\sqrt{2\xi}} v^2 dv = \frac{2/3}{B_o} \int_o^\infty (2\xi)^{3/2} g(\xi) d\xi$$

$$p_\perp^o = \frac{2}{B_o} \int_o^\infty g(\xi) d\xi \int_o^{\sqrt{2\xi}} (\xi - \tfrac{1}{2} v^2) dv = \frac{2/3}{B_o} \int_o^\infty (2\xi)^{3/2} g(\xi) d\xi \ .$$

Q.E.D.

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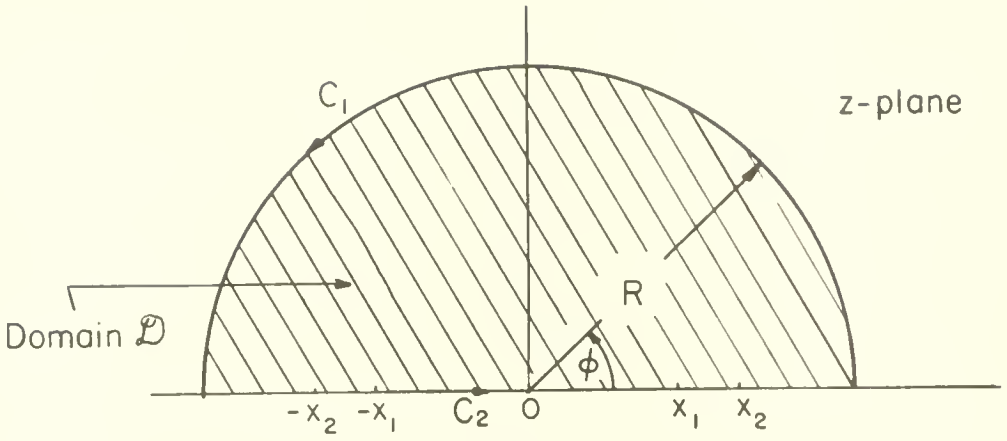

Figure 1

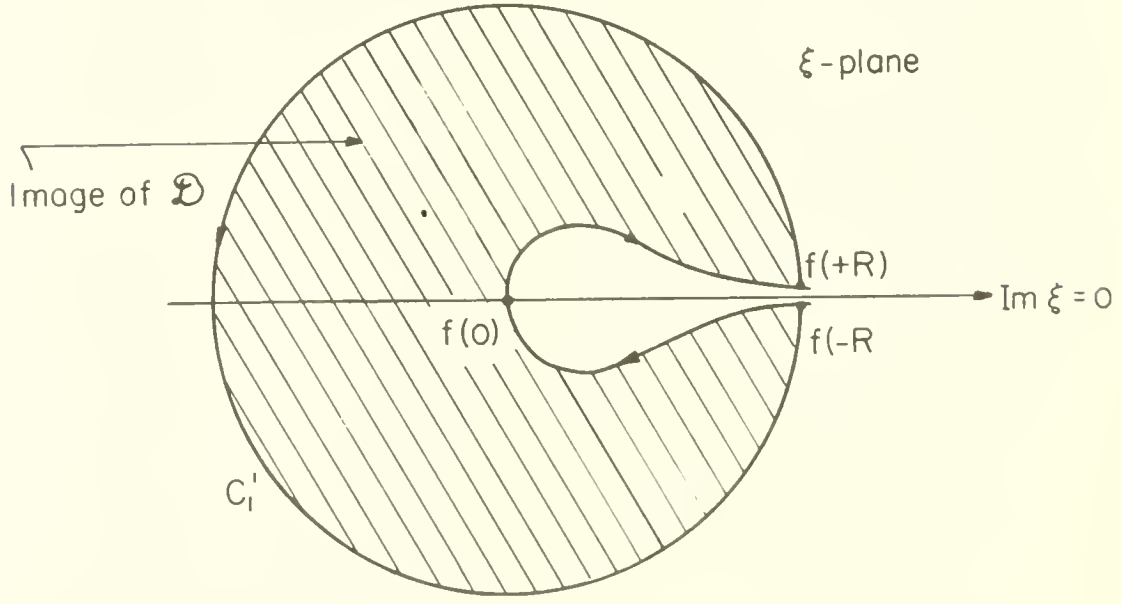

Figure 2
Illustrating a One-One Mapping of $\mathcal{D}$ by f(z)

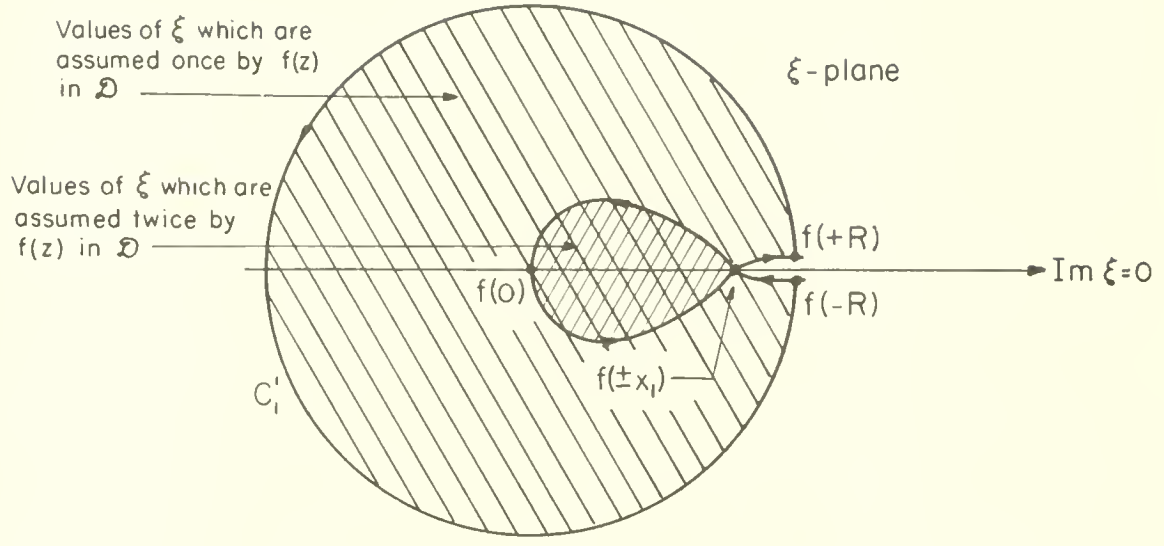

Figure 3
Illustrating a Mapping of $\mathcal{D}$ by f(z) Which is not One-One

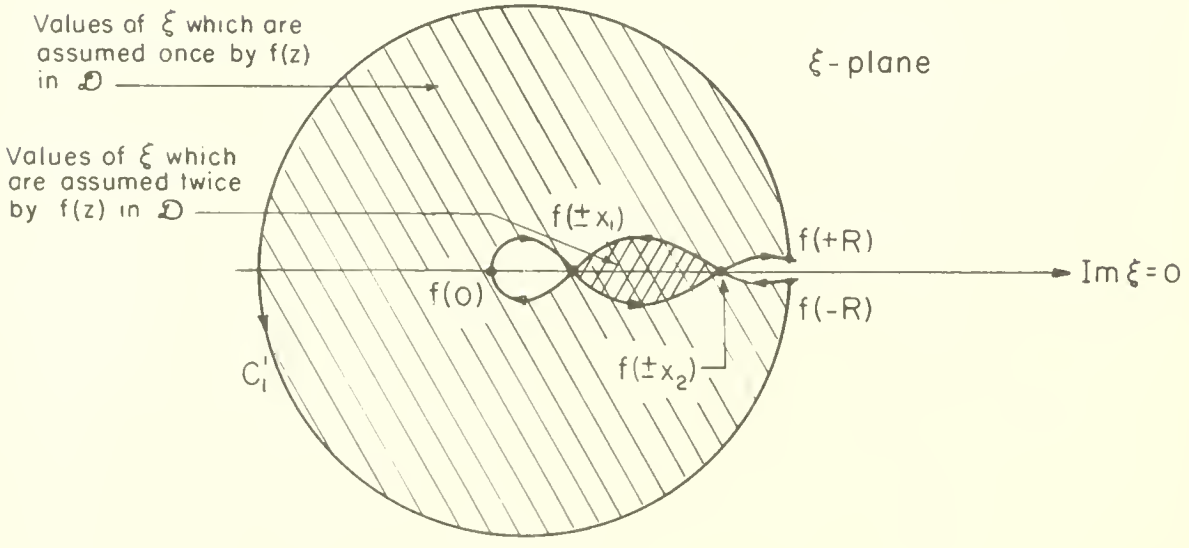

Figure 4

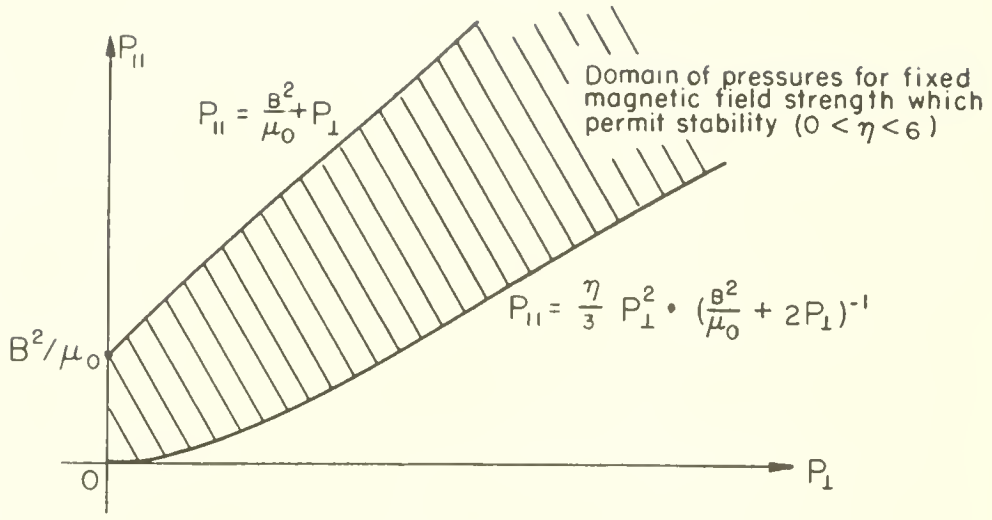
P_{11}
$P_{11} = \frac{B^2}{\mu_0} + P_\perp$
Domain of pressures for fixed magnetic field strength which permit stability $(0 < \eta < 6)$
$P_{11} = \frac{\eta}{3} P_\perp^2 \cdot (\frac{B^2}{\mu_0} + 2P_\perp)^{-1}$
B^2/μ_0
0
$P_\perp$

Figure 5

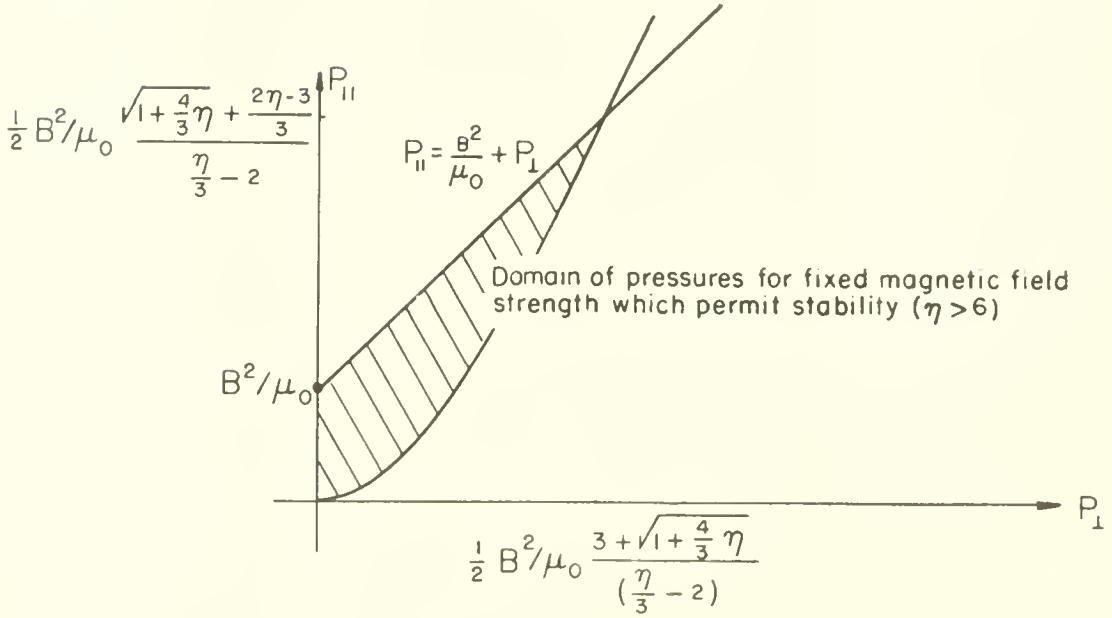
$\frac{1}{2} B^2/\mu_0 \frac{\sqrt{1+\frac{4}{3}\eta} + \frac{2\eta-3}{3}}{\frac{\eta}{3} - 2}$
P_{11}
$P_{11} = \frac{B^2}{\mu_0} + P_\perp$
Domain of pressures for fixed magnetic field strength which permit stability $(\eta > 6)$
B^2/μ_0
$P_\perp$
$\frac{1}{2} B^2/\mu_0 \frac{3+\sqrt{1+\frac{4}{3}\eta}}{(\frac{\eta}{3} - 2)}$

Figure 6

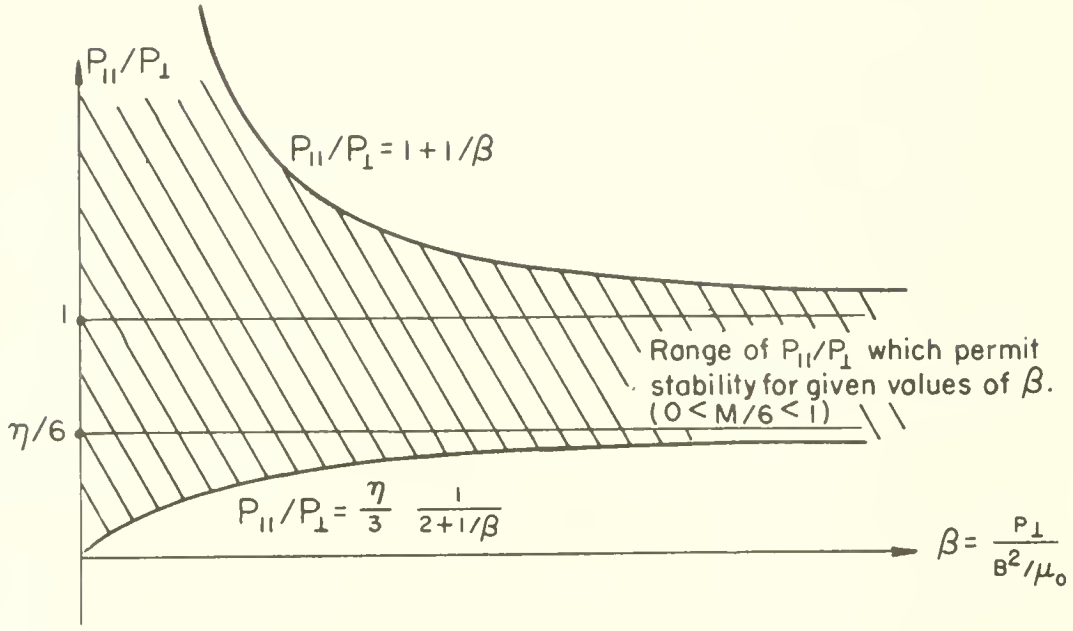
$P_{11}/P_\perp$
$P_{11}/P_\perp = 1 + 1/\beta$
1
$\eta/6$
Range of $P_{11}/P_\perp$ which permit stability for given values of β. $(0 < M/6 < 1)$
$P_{11}/P_\perp = \frac{\eta}{3} \frac{1}{2+1/\beta}$
$\beta = \frac{P_\perp}{B^2/\mu_0}$

Figure 7

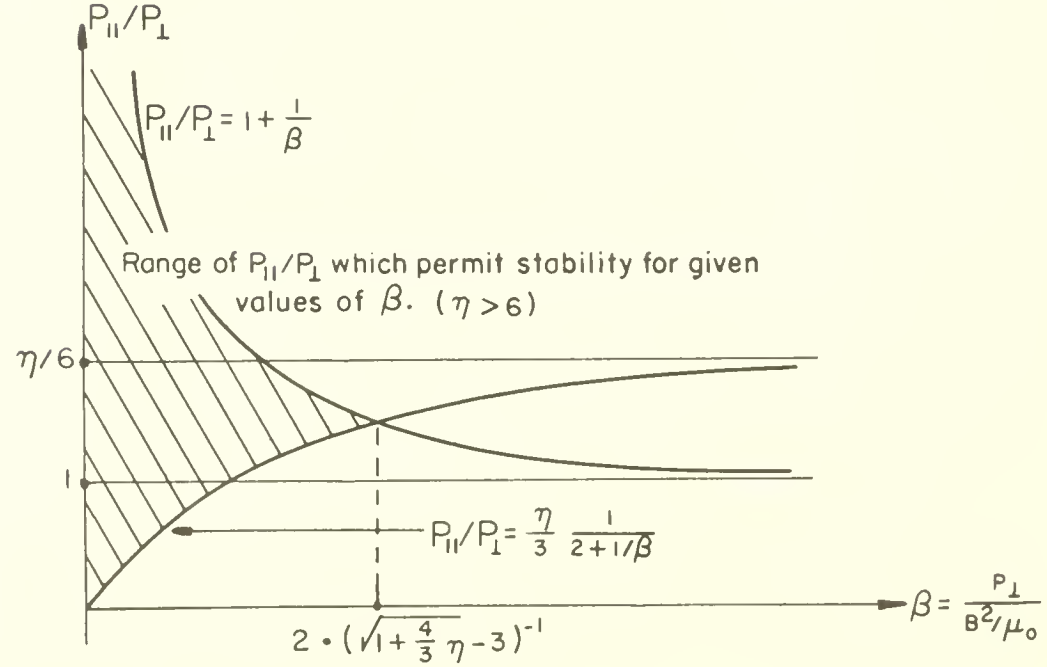
$P_{11}/P_\perp$
$P_{11}/P_\perp = 1 + \frac{1}{\beta}$
Range of $P_{11}/P_\perp$ which permit stability for given values of β. $(\eta > 6)$
$\eta/6$
1
$P_{11}/P_\perp = \frac{\eta}{3} \frac{1}{2+1/\beta}$
$\beta = \frac{P_\perp}{B^2/\mu_0}$
$2 \cdot (\sqrt{1+\frac{4}{3}\eta} - 3)^{-1}$

Figure 8

Illustrating Existence of Maximal β Beyond Which Stability is Impossible if $\eta > 6$

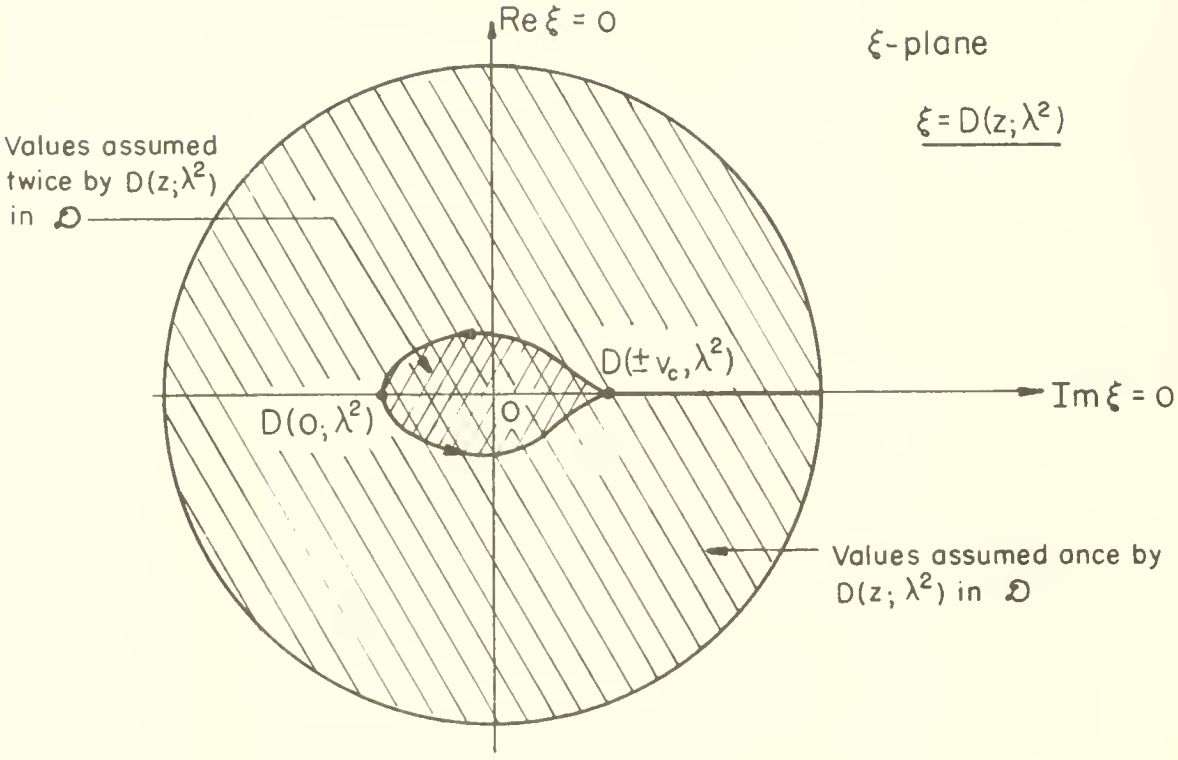

Figure 9

Illustrating an Impossible Mapping Induced by Cut-Off Functions Monotone in v^2

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